2011 Preliminary Examination in Calculus
UC Merced, Graduate Program in Applied Mathematics

You have four hours to complete this exam. There are ten problems total, each worth ten points. Show all work on separate sheets of paper, and circle your final answers, where appropriate. When finished, please staple your work behind these sheets, with your name written at the top. You are allowed one hand-written crib sheet, but no calculators or other study aides are allowed. Remember to explain your work clearly and legibly, so that you may receive maximum credit. Good luck!

Problem 1. Evaluate the following limits:

$$1.\ \lim_{x\to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x-3} \qquad 2.\ \lim_{x\to 0^+} (1+x)^{4/x} \qquad 3.\ \lim_{t\to 0} \frac{\tan^{-1} t}{\sin^{-1} t}$$

Problem 2. For each of the following integrals, determine whether the integral converges. If it does, find its value.

$$1.\ \int_0^{\pi/4} \frac{\sec^2 x}{\sqrt{\tan x}}\,dx \qquad 2.\ \int_0^1 \frac{2x+3}{x^2(x-2)}\,dx \qquad 3.\ \int_0^{+\infty} e^{-x}\sin x\,dx$$

Problem 3. Explain why the function

$$f(x) = \frac{1}{\sin x} - \frac{1}{x}, \qquad 0 < x \le \frac{\pi}{2}$$

is positive and increasing.

Problem 4. Express

$$\int_0^1 x^{1/5} e^x\,dx$$

as an infinite series.

Problem 5. If x thousand dollars is spent on labor, and y thousand dollars is spent on equipment it is estimated that the output of a certain factory will be

$$Q(x,y) = 50x^{2/5}y^{3/5}$$

units. If \$150,000 is available, how should this capital be allocated between labor and equipment to generate the largest possible output? How does the maximum output change if the money available for labor and equipment is increased by \$1,000?

Problem 6. Evaluate

$$\oint_C [(1+y)z\,dx + (1+z)x\,dy + (1+x)y\,dz],$$

where C is any closed path in the plane $2x-3y+z=1$.

Problem 7. The energy required to compress a gas from pressure p_1 to pressure p_{N+1} in N stages is proportional to

$$E = \left(\frac{p_2}{p_1}\right)^2 + \left(\frac{p_3}{p_2}\right)^2 + \cdots + \left(\frac{p_{N+1}}{p_N}\right)^2 - N.$$

Show how to choose the intermediate pressures $p_2, \ldots, p_N$ so as to minimize the energy requirement.

Problem 8. Let $\mathbf{F}$ be the following vector field in $\mathbb{R}^2$:

$$\mathbf{F} = [(2x - x^2 y)e^{-xy} + \tan^{-1} y]\mathbf{i} + \left[\frac{x}{y^2+1} - x^3 e^{-xy}\right]\mathbf{j}.$$

(a) Without explicitly finding a potential function for $\mathbf{F}$, explain why $\mathbf{F}$ is conservative.

(b) Using the result of (a), explain the quickest way of evaluating $\oint_{C_1} \mathbf{F} \cdot d\mathbf{R}$, where C_1 is the ellipse $9x^2 + 4y^2 = 36$.

(c) Now evaluate $\oint_{C_2} \mathbf{F} \cdot d\mathbf{R}$, where C_2 is the curve with parametric equations $x = t^2 \cos(\pi t)$, $y = e^{-t} \sin(\pi t)$, and $0 \leq t \leq 1$.

Problem 9. A vector field $\mathbf{F}$ is *incompressible* in a region $D \subset \mathbb{R}^3$ if $\operatorname{div} \mathbf{F} = 0$ everywhere in D. If $\mathbf{F}$ and $\mathbf{G}$ are both conservative vector fields in D, show that $\mathbf{F} \times \mathbf{G}$ is incompressible.

Problem 10. Use Taylor series to find a non-trivial, approximate solution ϕ of

$$\sin\phi + b(1 + \cos^2\phi + \cos\phi) = 0,$$

where b is a small, positive constant. Your answer should depend on b.