Directions: This examination lasts 4 hours. Only one cheat sheet is allowed, but no electronic devices of any kind are permitted. Partial credit will be awarded to any relevant work; no credit will be awarded for unexplained answers, correct or not. Computations mistakes will be very lightly penalized. Accurate graphic representation of a problem will receive high consideration.

Throughout this exam, complex numbers are represented as $z = x + iy$, with $x$ and $y$ real numbers, and complex functions are denoted by $f(z) = u(x, y) + iv(x, y)$, where $u$ and $v$ are real functions of two real variables.

1. Find all complex numbers $z$ such that the following equalities hold and indicate where these points are on the complex plane.
   
   (a) $z^4 = -16$
   
   (b) $e^z = 2 + 2i$
   
   (c) $\bar{z} = -z$
   
   (d) $\arcsin(z) = \pi i$

2. Consider a complex function $f(z)$ and a point in the complex plane $z_0$
   
   (a) Using the definition of the derivative, find an expression for $f'(z_0)$ by taking a limit along the $x$-direction.
   
   (b) Find another expression for $f'(z_0)$ by taking a limit along the $y$-direction.
   
   (c) If $f(z)$ is differentiable at $z_0$, what is the relation between the two expression you found? What is this called?

3. Find the inverse of the function $w = f(z) = e^{iz} \cos z$.

4. Consider the contour $C$ made of the portion of the parabola $x = 9 - y^2$ that lies to the right of the $y$-axis, starting at $(0, 3)$ and ending at $(0, -3)$. Integrate the complex function $f(z) = ze^z$ along this contour.

5. Series
   
   (a) Determine the Taylor Series and its radius of convergence of $f(z) = 1/z$ centered at $z = -i$.
   
   (b) Determine the Laurent Series centered at $z = 1$ valid for $1 < |z - 1|$ of $f(z) = e^{z-1} + 2/z$.

6. Evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+9)^2} dx$ using Cauchy’s Residue Theorem.

7. Evaluate $\int_{0}^{\infty} \frac{4}{x^{3/4}(x^2+1)} dx$ using Cauchy’s Residue Theorem.
8. (a) Sketch what happens to the inside, outside and boundary of the unit circle under the map 
\[ w = f(z) = \frac{1}{z} + 2 - i. \]
(b) Sketch how the map \( f(z) = e^z \) acts on the square having vortices \((0, 0), (\pi/4, 0), (0, \pi/4)\) and \((\pi/4, \pi/4)\).

9. Describe, in words and with a figure, what you would need to use the Poisson formula given below to solve Laplace’s equation on a simply closed domain other than the unit circle. You do not have to include every detail, but **everything you write must make mathematical sense**.

\[
u(r, \theta) = \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - \phi) - r^2} u(1, \phi) \, d\phi \quad (1)
\]

10. Consider the domain \( D \) in the \( xy \)-plane, enclosed by:
   I - the \( x \)-axis between 0 and 1,
   II - the hyperbola \( y = (x^2 - 1)^{1/2} \) for \( x \) between 1 and \( \sqrt{\phi} \),
   III - the hyperbola \( y = \frac{1}{x} \) for \( x \) between 1 and \( \sqrt{\phi} \),
   IV - the line \( y = x \) for \( x \) between 0 and 1,
   where \( \phi = \frac{1 + \sqrt{5}}{2} \).

   (a) Draw the domain.
   (b) Using a complex mapping, solve Laplace’s equation on \( D \) subject to the conditions that
   \( f(x, y) = 2x^2 \) along I,
   \( f(x, y) = 2 \) along II,
   \( f(x, y) = 2x^2 - 2/x^2 \) along III,
   and \( f(x, y) = 0 \) along IV.
   (Hint: map the domain to a rectangle).