Graduate level preliminary exam: **Complex Variables** **UC Merced, January 2009**

Directions: This examination lasts 4 hours. Only one cheat sheet is allowed, but no electronic devices of any kind are permitted. Partial credit will be awarded to any relevant work; no credit will be awarded for unexplained answers, correct or not. Computations mistakes will be very lightly penalized. Accurate graphic representation of a problem will receive high consideration.

Throughout this exam, complex numbers are represented as $z = x + iy$, with x and y real numbers, and complex functions are denoted by $f(z) = u(x, y) + iv(x, y)$, where u and v are real functions of two real variables.

1. Find all complex numbers z such that the following equalities hold and indicate where these points are on the complex plane.
   (a) $z^4 = -16$
   (b) $e^z = 2 + 2i$
   (c) $\bar{z} = -z$
   (d) $\arcsin(z) = \pi i$
2. Consider a complex function $f(z)$ and a point in the complex plane z_0
   (a) Using the definition of the derivative, find an expression for $f'(z_0)$ by taking a limit along the x-direction.
   (b) Find another expression for $f'(z_0)$ by taking a limit along the y-direction.
   (c) If $f(z)$ is differentiable at z_0, what is the relation between the two expression you found? What is this called?
3. Find the inverse of the function $w = f(z) = e^{iz} \cos z$.
4. Consider the contour C made of the portion of the parabola $x = 9 - y^2$ that lies to the right of the y-axis, starting at $(0, 3)$ and ending at $(0, -3)$. Integrate the complex function $f(z) = ze^z$ along this contour.
5. Series
   (a) Determine the Taylor Series and its radius of convergence of $f(z) = 1/z$ centered at $z = -i$.
   (b) Determine the Laurent Series centered at $z = 1$ valid for $1 < |z - 1|$ of $f(z) = e^{z-1} + 2/z$.
6. Evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+9)^2} dx$ using Cauchy's Residue Theorem.
7. Evaluate $\int_0^{\infty} \frac{4}{x^{1/4}(x^2+1)} dx$ using Cauchy's Residue Theorem.

8. (a) Sketch what happens to the inside, outside and boundary of the unit circle under the map $w = f(z) = 1/z + 2 - i$.

   (b) Sketch how the map $f(z) = e^z$ acts on the square having vortices $(0,0)$, $(\pi/4, 0)$, $(0, \pi/4)$ and $(\pi/4, \pi/4)$

9. Describe, in words and with a figure, what you would need to use the Poisson formula given below to solve Laplace's equation on a simply closed domain other than the unit circle. You do not have to include every detail, but **everything you write must make mathematical sense**.

$$u(r,\theta) = \int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)-r^2} u(1,\phi)\, d\phi \tag{1}$$

10. Consider the domain D in the xy-plane, enclosed by:
    I - the x-axis between 0 and 1,
    II - the hyperbola $y = (x^2-1)^{1/2}$ for x between 1 and $\sqrt{\phi}$,
    III - the hyperbola $y = \frac{1}{x}$ for x between 1 and $\sqrt{\phi}$
    IV - the line $y = x$ for x between 0 and 1,
    where $\phi = \frac{1+\sqrt{5}}{2}$.

    (a) Draw the domain.

    (b) Using a complex mapping, solve Laplace's equation on D subject to the conditions that
    $f(x,y) = 2x^2$ along I,
    $f(x,y) = 2$ along II,
    $f(x,y) = 2x^2 - 2/x^2$ along III,
    and $f(x,y) = 0$ along IV.
    (Hint: map the domain to a rectangle).