Directions: This examination lasts 4 hours. One hand-written sheet of notes is allowed, but no electronic device of any kind is permitted. Partial credit will be awarded to any relevant work; no credit will be awarded for unexplained answers, correct or not. Computational mistakes will be very lightly penalized. Accurate graphic representation of a problem will receive high consideration. The final grade will be computed out of 90.

1. (12 pts) Find all complex numbers z such that the following equalities hold and indicate where these points are on the complex plane.
   (a) $z^3 = -64i$
   (b) i^{1+i}
   (c) $\bar{\bar{\bar{z}}} = z$
2. (10 pts) Using the quadratic formula, express the function $w = f(z) = \text{Arccos}\, z$ in terms of elementary functions. You should start from the form $\cos w = z$.
3. (12 pts) Explain why the following statements are true:
   (a) $\int_{C_1} e^{1/z} dz = \int_{C_2} e^{1/z} dz$ with C_1 and C_2 circles centered at $z = 3$ of radii 1 and 2, respectively.
   (b) if $f(z)$ is analytic, then there exists a function $H(z)$ such that $\int_{z_1}^{z_2} e^{f(z)} dz = H(z_2) - H(z_1)$.
   (c) Let $f(z) = \frac{\sin z - z}{z^3}$ if $z \neq 0$ and $f(0) = -1/6$, then $f(z)$ is analytic at the origin.
4. (10 pts) Determine the Taylor Series centered at $z = 2i$ of $f(z) = \frac{1}{z-i}$ and determine its radius of convergence. From the Taylor Series, find $f^{(5)}(2i)$.
5. (20 pts) Consider the function $G(z) = \frac{\cos(\pi z)}{z^2 \sin(\pi z)}$. You may use the identities:
   $\cos(z) = \cos x \cosh y - i \sin x \sinh y$
   $\sin(z) = \sin x \cosh y + i \cos x \sinh y$.
   (a) Find and classify all the singularities of $G(z)$.
   (b) Find the residues at all the singularities of $G(z)$ (you may use the table provided at the end of the exam).
   (c) Consider the contour ✠ made of a square centered at the origin, extending from $-\pi(n + 1/2)$ to $\pi(n + 1/2)$, with $n \in \mathbb{N}$, in both the x and y direction. Bound $\int_{✠} G(z) dz$ as a function of n.
   (d) Take the limit as $n \to \infty$ and use the Residue theorem, to obtain a famous identity.
6. (10 pts) Evaluate $\int_0^\infty \frac{dx}{x^{1/2}(x^2-4x+8)}$ using Cauchy's Residue Theorem and the so-called pacman integration contour.
7. (8 pts) Describe the following transforms of the z-plane onto the w-plane in words and by mapping relevant sub-domains.
   (a) $w = f(z) = iz^2$
   (b) $w = f(z) = \log(z - 1)$

8. (8 pts) Give simple mappings that accomplish the following transformations of the plane
   (a) Transforms the circle $z = i + 2e^{i\theta}$ into the unit circle centered at the origin, inverting the inside and outside of the circle.
   (b) Translates the plane by two units downward and rotates the plane by $\pi/3$ counter-clockwise.

9. (10 pts) Using the complex mapping $f(z) = \frac{z-2i}{z}$, find the steady temperature distribution, a harmonic function, in the space between the circles $|z - i| = 1$ and $|z - 2i| = 2$, except at the origin. The temperature is maintained at 0 on the small circle and 3 on the large circle. Express your solution in cartesian coordinates, i.e. find $T(x, y)$.

$$\frac{d}{dz}\left(\frac{\cos(\pi z)}{z\sin(\pi z)}\right) = -(\cot(\pi z)/z^2) - (\pi\csc^2(\pi z))/z$$

$$\frac{d^2}{dz^2}\left(\frac{\cos(\pi z)}{z\sin(\pi z)}\right) = 2\cot(\pi z))/z^3 + (2\pi\csc^2(\pi z))/z^2 + (2\pi^2\cot(\pi z)\csc^2(\pi z))/z$$

$$\frac{d^3}{dz^3}\left(\frac{\cos(\pi z)}{z\sin(\pi z)}\right) = (-6\cot(\pi z))/z^4 - (6\pi\csc^2(\pi z))/z^3 - (6\pi^2\cot(\pi z)\csc^2(\pi z))/z^2 - (4\pi^3\cot(\pi z)^2\csc^2(\pi z))/z - (2\pi^3\csc^4(\pi z))/z$$

$$\frac{d}{dz}\left(\frac{\cos(\pi z)}{\sin(\pi z)}\right) = -(\pi\csc^2(\pi z))$$

$$\frac{d^2}{dz^2}\left(\frac{\cos(\pi z)}{\sin(\pi z)}\right) = 2\pi^2\cot(\pi z)\csc^2(\pi z)$$

$$\frac{d^3}{dz^3}\left(\frac{\cos(\pi z)}{\sin(\pi z)}\right) = -4\pi^3\cot^2(\pi z)\csc^2(\pi z) - 2\pi^3\csc^4(\pi z)$$

$$\frac{d}{dz}\left(\frac{z\cos(\pi z)}{\sin(\pi z)}\right) = \cot(\pi z) - \pi z\csc^2(\pi z)$$

$$\frac{d^2}{dz^2}\left(\frac{z\cos(\pi z)}{\sin(\pi z)}\right) = -2\pi\csc^2(\pi z) + 2\pi^2 z\cot(\pi z)\csc^2(\pi z)$$

$$\frac{d^3}{dz^3}\left(\frac{z\cos(\pi z)}{\sin(\pi z)}\right) = 6\pi^2\cot(\pi z)\csc^2(\pi z) - 4\pi^3 z\cot(\pi z)^2\csc^2(\pi z) - 2\pi^3 z\csc^4(\pi z)$$

$$\frac{d}{dz}\left(\frac{z^2\cos(\pi z)}{\sin(\pi z)}\right) = 2z\cot(\pi z) - \pi z^2\csc^2(\pi z)$$

$$\frac{d^2}{dz^2}\left(\frac{z^2\cos(\pi z)}{\sin(\pi z)}\right) = 2\cot(\pi z) - 4\pi z\csc^2(\pi z) + 2\pi^2 z^2\cot(\pi z)\csc^2(\pi z)$$

$$\frac{d^3}{dz^3}\left(\frac{z^2\cos(\pi z)}{\sin(\pi z)}\right) = -6\pi\csc^2(\pi z) + 12\pi^2 z\cot(\pi z)\csc^2(\pi z) - 4\pi^3 z^2\cot^2(\pi z)\csc^2(\pi z) - 2\pi^3 z^2\csc^4(\pi z)$$