Duration: 240 minutes
Instructions: Answer all questions. You may use one cheat-sheet. Partial credit will be awarded for correct work, unless otherwise specified. The total number of points is 100.

1. Compute AND plot the following complex numbers
   (a) $z = \log(\sqrt{3} + i)$
   (b) $\sin z = 2$
   (c) $z^3 = 1 + i$

2. (21 pts) In the following proof of the Cauchy Integral formula, explain why each of the seven numbered statements hold. Your explanations should be brief (one sentence or formula for each statement).

   Thm: If $f(z)$ is analytic on and inside the closed contour C and z_0 is inside C, then

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz - f(z_0) = 0.$$

   Proof: Let C_ρ be a circle of radius ρ centered at z_0, with ρ sufficiently small so that C_ρ is entirely inside C.

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz - f(z_0) = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)}{z-z_0}dz - f(z_0) = 0 \tag{1}$$

$$\frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)}{z-z_0}dz - f(z_0) = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}dz \tag{2}$$

   Also, $\forall \epsilon > 0, \exists \rho > 0$ such that $|z - z_0| \le \rho \longrightarrow |f(z) - f(z_0)| < \epsilon$, so (3)

$$\left|\frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}dz\right| \le \frac{1}{2\pi}\int_{C_\rho} \left|\frac{f(z)-f(z_0)}{z-z_0}\right| |dz| \tag{4}$$

$$\frac{1}{2\pi}\int_{C_\rho} \left|\frac{f(z)-f(z_0)}{z-z_0}\right| |dz| < \frac{1}{2\pi}\int_{C_\rho} \left|\frac{\epsilon}{\rho}\right| |dz| \tag{5}$$

$$\frac{1}{2\pi}\int_{C_\rho} \left|\frac{\epsilon}{\rho}\right| |dz| \le \epsilon \tag{6}$$

   Therefore $\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz - f(z_0) = 0.$ (7)

3. Consider the function $f(z) = e^z + |z|^2 + \frac{z^2 - \bar{z}^2}{2}$.
   (a) Determine where this function is differentiable.
   (b) What is $f'(z)$ when the function is differentiable?

4. Compute $\int_C f(z)dz$ for the following functions and paths C:

(a) $f(z) = ze^z + i(Re(z))^2 + z$ over the line segment going from i to 2.
(b) $f(z) = \frac{\sin(z)}{z^2+4}$ over the circle of radius 2 centered at $1 - i$.

5. Compute Laurent Series that are valid over the prescribed regions for the following functions
   (a) $f(z) = \frac{\log(1+z)}{1-z}$ over $|z| < 1$.
   (b) $f(z) = \frac{1}{2z^2+2iz+12}$ over $2 < |z| < 3$.

6. Compute the residues at all the isolated singularities of the following functions.
   (a) $f(z) = (\tanh z)^2$
   (b) $f(z) = \frac{1}{z^{2/3}+z^{8/3}}$.

7. Consider the function $f(R) = \int_{C_R} \frac{dz}{(1+z^2)(z^2-z-6)}$, where C_R is the upper half-circle of radius R centered at the origin with diameter on the real axis (positively oriented). Compute $f(R)$ for $R > 0$.

8. Compute $\int_0^\infty \frac{dx}{x^a(2+x)}$, with $a \in \mathbb{R}$, using the Residue Theorem, and provide a range for values of a for which this procedure is valid.

9. Describe the effects of the following mappings on representative curves within the complex planes
   (a) $f(z) = z^4$
   (b) $f(z) = \frac{1+z}{i-z}$
   (c) $f(z) = e^z$

10. Use a polynomial complex mapping to solve Laplace's equation for $h(x, y)$ over the region bounded by the four sides $C_1 : 0 \le x \le 1$, $C_2 : y = x$, for $0 \le x \le 1$, $C_3 : x^2 - y^2 = 1$, for $1 \le x \le \sqrt{\frac{1+\sqrt{5}}{2}}$, and $C_4 : y = 1/x$ for $1 \le x \le \sqrt{\frac{1+\sqrt{5}}{2}}$. with the following boundary conditions:
    $\frac{\partial h}{\partial n} = 0$ along C_1 and C_4,
    $h(x, y) = 2$ along C_2,
    $h(x, y) = 7$ along C_3.