1. Compute AND plot the following complex numbers
   (a) \( z = \log(\sqrt{3} + i) \)
   (b) \( \sin z = 2 \)
   (c) \( z^3 = 1 + i \)

2. (21 pts) In the following proof of the Cauchy Integral formula, explain why each of the seven numbered statements hold. Your explanations should be brief (one sentence or formula for each statement).

Thm: If \( f(z) \) is analytic on and inside the closed contour \( C \) and \( z_0 \) is inside \( C \), then
\[
\frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz - f(z_0) = 0.
\]

Proof: Let \( C_\rho \) be a circle of radius \( \rho \) centered at \( z_0 \), with \( \rho \) sufficiently small so that \( C_\rho \) is entirely inside \( C \).

\[
\frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz - f(z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z - z_0} dz - f(z_0) = 0 \tag{1}
\]

\[
\frac{1}{2\pi i} \oint_{C_\rho} \frac{f(z)}{z - z_0} dz - f(z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz \tag{2}
\]

Also, \( \forall \epsilon > 0, \exists \rho > 0 \) such that \( |z - z_0| \leq \rho \implies |f(z) - f(z_0)| < \epsilon \), so
\[
\left| \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq \frac{1}{2\pi} \int_{C_\rho} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| |dz| \tag{3}
\]

\[
\frac{1}{2\pi} \int_{C_\rho} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| |dz| < \frac{1}{2\pi} \int_{C_\rho} \frac{\epsilon}{\rho} |dz| \tag{4}
\]

\[
\frac{1}{2\pi} \int_{C_\rho} \frac{\epsilon}{\rho} |dz| \leq \epsilon \tag{5}
\]

Therefore
\[
\frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz - f(z_0) = 0. \tag{7}
\]

3. Consider the function \( f(z) = e^z + |z|^2 + \frac{z^2 - \bar{z}^2}{2} \).
   (a) Determine where this function is differentiable.
   (b) What is \( f'(z) \) when the function is differentiable?

4. Compute \( \oint_C f(z) dz \) for the following functions and paths \( C \):
5. Compute Laurent Series that are valid over the prescribed regions for the following functions
   (a) \( f(z) = \frac{\log(1+z)}{1-z} \) over \(|z| < 1\).
   (b) \( f(z) = \frac{1}{2z^4 + 2iz + 12} \) over \(2 < |z| < 3\).

6. Compute the residues at all the isolated singularities of the following functions.
   (a) \( f(z) = (\tanh z)^2 \)
   (b) \( f(z) = \frac{1}{z/(2+z)} \).

7. Consider the function \( f(R) = \int_{C_R} \frac{dz}{(1+z^2)(2x-z-6)} \), where \( C_R \) is the upper half-circle of radius \( R \) centered at the origin with diameter on the real axis (positively oriented). Compute \( f(R) \) for \( R > 0 \).

8. Compute \( \int_0^\infty \frac{dx}{x^{a+(2+x)}} \), with \( a \in \mathbb{R} \), using the Residue Theorem, and provide a range for values of \( a \) for which this procedure is valid.

9. Describe the effects of the following mappings on representative curves within the complex planes
   (a) \( f(z) = z^4 \)
   (b) \( f(z) = \frac{1+z}{i-z} \)
   (c) \( f(z) = e^z \)

10. Use a polynomial complex mapping to solve Laplace’s equation for \( h(x,y) \) over the region bounded by the four sides \( C_1 : 0 \leq x \leq 1, C_2 : y = x, \) for \( 0 \leq x \leq 1, C_3 : x^2 - y^2 = 1, \) for \( 1 \leq x \leq \sqrt{1+\sqrt{5}}/2 \), and \( C_4 : y = 1/x \) for \( 1 \leq x \leq \sqrt{1+\sqrt{5}}/2 \). with the following boundary conditions:
    \( \frac{\partial h}{\partial n} = 0 \) along \( C_1 \) and \( C_4, \)
    \( h(x, y) = 2 \) along \( C_2, \)
    \( h(x, y) = 7 \) along \( C_3. \)