Ph.D. Candidates Preliminary Exam: Linear Algebra

UC Merced, January 2007

Directions: This examination lasts 4 hours.

1) Determine the rank of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}.$$

2) Reduce the following matrix A to echelon form and use it to find *all* solutions of the system

$$Ax = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix} = b.$$

3) (a) Let V and W be vector spaces over F. Give the definition of a *linear transformation* $L : V \to W$.
   (b) Define the *null space* of L and prove it is a subspace of V.
   (c) Define the *image* of L - also called the *range* of L - and prove that it is a subspace of W.

4) For what values of a is the following matrix positive-definite:

$$A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$$

5) Suppose A is symmetric positive definite and Q is an orthogonal matrix. Determine whether the following statements are true or false:
   (a) $Q^T AQ$ is a diagonal matrix,
   (b) $Q^T AQ$ is symmetric positive definite,
   (c) $Q^T AQ$ has the same eigenvalues as A,
   (d) e^{-A} is symmetric positive definite.

6) If $u^H u = 1$, show that $I - 2uu^H$ is Hermitian and also unitary. The rank-1 matrix uu^H is the projection onto what line in C^n?

7) Show that the vectors of the following basis

$$x_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

are linearly independent and construct an equivalent orthogonal basis.

8) Find all values of α for which the following matrix is non-singular:

$$\begin{bmatrix} -2 & \alpha & 3 \\ 1 & 2 & \alpha \\ 1 & 11 & 18 \end{bmatrix}$$

9) Given $A = \begin{bmatrix} -3 & 2 & 1 \\ -7 & 6 & 5 \\ 2 & -2 & -2 \end{bmatrix}$, find a matrix P such that $D = P^{-1}AP$ is a diagonal matrix. What are the elements of the matrix D called?

10) Prove that similar matrices have the same characteristic polynomial and the same eigenvalues (A and B are called *similar* if there exist a non-singular matrix X such that $A = X^{-1}BX$).

11) Recall that projection matrices satisfy $P = P^T$ and $P^2 = P$. Show that the eigenvalues of a projection matrix are either zero or one.