Applied Math Preliminary Exam: Linear Algebra
University of California, Merced, January 2010

Instructions: This examination lasts 4 hours.

- Show explicitly steps and computations in your solutions. Credit will not be given to answers without explanation.
- Partial credit will be awarded to relevant work.

Problem 1. Given the system of linear equations

\[
\begin{align*}
-x_1 + x_2 + x_3 + x_4 &= b_1 \\
-2x_1 + x_2 - 3x_3 - x_4 &= b_2 \\
-2x_1 + 2x_2 - x_3 - x_4 &= b_3
\end{align*}
\]

a. (5 points) find all possible values of \(b_1, b_2,\) and \(b_3\) for which this system has solutions;

b. (5 points) find all possible solutions of this system if \(b_1 = -10,\) \(b_2 = 8,\) and \(b_3 = 18.\)

Problem 2. (10 points) Let \(A \in \mathbb{R}^{4 \times 4}\) be a product of two matrices given by

\[
A = \begin{bmatrix}
1 & -1 & -2 \\
1 & 1 & 2 \\
1 & 0 & -1 \\
1 & 0 & -1
\end{bmatrix} \begin{bmatrix}
3 & 2 & 1 & 0 \\
0 & 2 & 0 & 0 \\
-1 & 2 & 0 & 0
\end{bmatrix}.
\]

Find dimensions and bases for \(\text{Range}(A), \text{Null}(A), \text{Range}(A^T),\) and \(\text{Null}(A^T).\)

Problem 3. (10 points) Prove that if \(A \in \mathbb{R}^{n \times n}\) is symmetric, positive definite, and orthogonal, then \(A\) must be the identity matrix. (Hint: Consider the eigenvalues of \(A.\))

Problem 4. Let \(x \in \mathbb{R}^n\) and \(A = uu^T\) where \(u, v \in \mathbb{R}^n.\)

a. (5 points) Prove that \(\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1.\)

b. (5 points) Prove that \(\|A\|_2 = \|u\|_2 \|v\|_2.\)

Problem 5. Let \(H\) be the Householder transformation given by

\[
H = I - \frac{2}{\|u\|_2^2} uu^T,
\]

where \(u \in \mathbb{R}^n.\)

a. (5 points) Show that \(H\) is symmetric and orthogonal.

b. (5 points) Determine the eigenvalues and the corresponding eigenvectors of \(H.\)

c. (5 points) Let \(x \in \mathbb{R}^n\) have unit length, i.e., \(\|x\|_2 = 1,\) and let \(e_1\) be the first column of the \(n \times n\) identity matrix, i.e., \(e_1 = (1, 0, 0, \cdots, 0)^T.\) Show that if \(u = x - e_1,\) then

\[
Hx = e_1.
\]
Problem 6. Solve the least-squares problem

\[ \min_{x \in \mathbb{R}^3} \|Ax - b\|_2 \]

where

\[ A = \begin{bmatrix} 2 & 7 \\ -1 & -5 \\ -2 & -4 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}. \]

using

a. (5 points) the normal equation,

b. (10 points) the QR factorization.

Verify that your solutions agree.

Problem 7. Let \( A \in \mathbb{R}^{m \times n} \) have rank \( n \).

a. (5 points) Show that \( P = A(A^T A)^{-1} A^T \) is a projection matrix.

b. (10 points) By considering the singular value decomposition of \( A \), show that

\[ \| A (A^T A)^{-1} A^T \|_2 = 1. \]

Problem 8. Give an example for each of the following. Briefly explain why.

a. (3 points) A matrix that is positive definite but not symmetric.

b. (3 points) A non-diagonalizable matrix.

Problem 9. State whether each of the following statements is true or false. Briefly explain why.

a. (3 points) If \( y \in \mathbb{R}^n \) is not in the null space of \( A \in \mathbb{R}^{m \times n} \), then it must be in the range space of \( A^T \).

b. (3 points) Any underdetermined linear system \( Ax = b \), where \( A \in \mathbb{R}^{m \times n} \) with \( m < n \) and \( b \in \mathbb{R}^m \), has infinitely many solutions.

c. (3 points) If \( A \) is an invertible matrix, then the inverse of its transpose is the same as the transpose of its inverse, i.e.,

\[ (A^T)^{-1} = (A^{-1})^T. \]