1. (20 points, 4 points each) Determine whether the following statements are True or False.

   (a) If an object moves with the same average velocity over every time interval, then its average velocity equals its instantaneous velocity at any time.

   (b) If $f'(x) = g'(x)$ for all real number x, then $f(x) = g(x)$.

   (c) The sinusoidal function $y = -3\sin(4x) + 5$ completes 4 cycles in the interval $[0, 2\pi]$.

   (d) For sufficiently large values of x, $f(x) = 1000x^3 + 345x^2 + 17x + 394$ is less than $g(x) = 0.01x^4$.

   (e) If $\lim_{x\to 3} f(x) = 7$ and $g(3) = 4$, then $\lim_{x\to 3}(f(x) + g(x)) = 11$.

   <u>Answers</u>: (a) True. (b) False. (c) True. (d) True. (e) False.

2. (30 points, 6 points each) Choose A, B, C, D, or E for each of the following questions.

   (a) Which of the following functions have an inverse?

   (I) $\cos x$ with domain $[0, 1]$ (II) $e^{-(x-1)}$ (III) $(x-2)^2$ with domain $(-\infty, 1]$

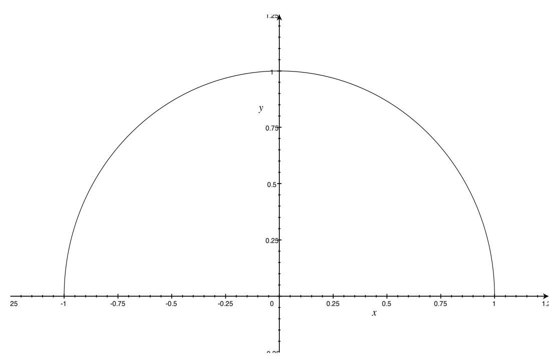
   A) II only **B)** I and II only **C)** I and III only **D)** III only **E)** I, II and III

   (b) Which of the following functions are increasing functions?

   (I) the derivative of an increasing function (II) the derivative of a concave up function
   (III) the inverse of an increasing function (IV) the inverse of a concave up function

   A) I and III only **B)** II and III only **C)** II and III only **D)** I and IV only **E)** III and IV only

   (c) The graph of a function $g(x)$ is given below. Which of the following statements about its derivative $g'(x)$ are true?



   (I) $g'(0) = 0$. (II) $g'(x)$ is an odd function.
   (III) $g'(x)$ is decreasing over $(-1, 1)$. (IV) $g'(x)$ has vertical asymptotes at $x = \pm 1$.

   A) I only **B)** I and II only **C)** I and III only **D)** II and III only **E)** I, II, III and IV

   (d) Which of the following statements are true?

   (I) If $f(x)$ is not continuous at $x = a$, then it is not differentiable at $x = a$.
   (II) If $f(x)$ is not differentiable at $x = a$, then it is not continuous at $x = a$.
   (III) If $f(x)$ is differentiable at $x = a$, then it is continuous at $x = a$.

   A) II only **B)** I and II only **C)** I and III only **D)** III only **E)** I, II and III

(e) Consider the logarithmic function $f(x) = c\ln(kx)$, where $c < 0$ and $k > 0$ are constants. The graph of $f(x)$ is
A) increasing and concave up. **B)** decreasing and concave up. **C)** increasing and concave down. **D)** decreasing and concave down.

Answers: (a) E. (b) B or C. (c) E. (d) C. (e) B.

3. (10 points) Consider the piecewise function $f(x)$ defined below. Can you find a value for b such that $f(x)$ is continuous at $x = 2$. If yes, find this value. If not, explain why.

$$f(x) = \begin{cases} \cos\left((x-1)\frac{\pi}{2}\right)\frac{x-2}{|x-2|}, & \text{for } x \neq 2 \\ b, & \text{for } x = 2. \end{cases}$$

Solutions: $f(x)$ is continuous at $x = 2$ if

$$b = f(2) = \lim_{x\to 2} f(x) = \lim_{x\to 2} \cos\left((x-1)\frac{\pi}{2}\right)\frac{x-2}{|x-2|}.$$

Because of the absolute value sign, we need to discuss two one–sided limits.

$$\lim_{x\to 2^+} \cos\left((x-1)\frac{\pi}{2}\right)\frac{x-2}{|x-2|} = \lim_{x\to 2^+} \cos\left((x-1)\frac{\pi}{2}\right)\frac{x-2}{x-2} = \lim_{x\to 2^+} \cos\left((x-1)\frac{\pi}{2}\right)$$
$$= \cos\left((2-1)\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0.$$

$$\lim_{x\to 2^-} \cos\left((x-1)\frac{\pi}{2}\right)\frac{x-2}{|x-2|} = \lim_{x\to 2^+} \cos\left((x-1)\frac{\pi}{2}\right)\frac{x-2}{-(x-2)} = \lim_{x\to 2^+} -\cos\left((x-1)\frac{\pi}{2}\right)$$
$$= -\cos\left((2-1)\frac{\pi}{2}\right) = -\cos\left(\frac{\pi}{2}\right) = -0 = 0.$$

Since left–hand limit and right–hand limit are equal,

$$\lim_{x\to 2} \cos\left((x-1)\frac{\pi}{2}\right)\frac{x-2}{|x-2|} = 0.$$

Therefore, when $b = 0$, $f(x)$ is continuous at $x = 2$.

4. (8 points) Use the Intermediate Value Theorem to show that the equation $e^x = x + 2$ has a solution on the interval [0,2].

Solutions: $f(x) = e^x - x - 2$ is continuous on $[0, 2]$. Because

$$f(0) = e^0 - 0 - 2 = 1 - 2 = -1 < 0, \quad \text{and}$$
$$f(2) = e^2 - 2 - 2 = e^2 - 4 > 0 \quad (\text{since } e > 2),$$

0 is a number between $f(0)$ and $f(2)$. By IVT, there exists a number c in $[0, 2]$ such that $f(c) = e^c - c - 2 = 0$. In other words,

$$e^c = c + 2,$$

or, c is a solution to the equation $e^x = x + 2$.

5. (10 points) $g(x) = \dfrac{1}{1-x}$. Using the definition of a derivative, find $g'(x)$.

Solutions:

$$\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{\frac{1}{1-(x+h)}-\frac{1}{1-x}}{h} = \lim_{h\to 0}\frac{1}{h}\frac{(1-x)-[1-(x+h)]}{[1-(x+h)](1-x)}\\
&= \lim_{h\to 0}\frac{1}{h}\frac{1-x-1+x+h}{[1-(x+h)](1-x)} = \lim_{h\to 0}\frac{1}{h}\frac{h}{[1-(x+h)](1-x)} = \lim_{h\to 0}\frac{1}{[1-(x+h)](1-x)}\\
&= \frac{1}{(1-x)^2}.
\end{aligned}$$

6. (10 points) What is the y–intercept of the tangent line to $m(x) = \dfrac{5x^3+1}{x}$ at $x=-1$?

Solutions: $m(x) = 5x^2 + \dfrac{1}{x} = 5x^2 + x^{-1} \Longrightarrow m'(x) = 10x - x^{-2} = 10x - \dfrac{1}{x^2}$. So the slope of the tangent line is

$$m'(-1) = 10(-1) - \frac{1}{(-1)^2} = -10 - 1 = -11.$$

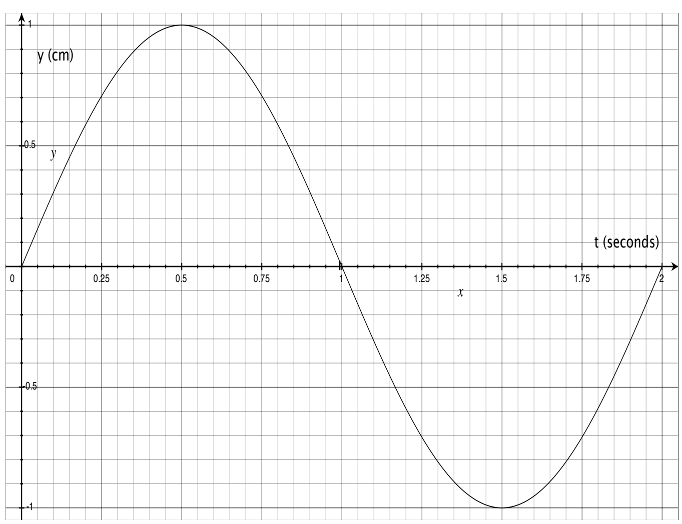
$$x = -1 \Longrightarrow y = m(-1) = 5(-1)^2 + \frac{1}{-1} = 5 - 1 = 4.$$

So a point on the tangent line is $(-1, 4)$. The equation of the tangent line is

$$\begin{aligned}
y - 4 &= -11(x+1)\\
y &= -11x - 11 + 4\\
y &= -11x - 7.
\end{aligned}$$

The y–intercept of the tangent line is -7.

7. A block attached to the end of a spring is moving vertically along the y–axis around $y=0$. The graph below shows its y–coordinate as a function of time t.



(a) (3 points) When (over what time interval(s)) is this block above $y=0$?

(b) (3 points) When (over what time interval(s)) is this block moving upward?

(c) (6 points) Is $\left.\dfrac{d^2y}{dt^2}\right|_{t=0.5}$ positive or negative? What are its units? What is its practical meaning?

Solutions:

(a) When $0 < t < 1$, the y–coordinate is positive, so the block stays above $y = 0$.

(b) When $0 \le t < 0.5$ and when $1.5 < t \le 2$, the y–coordinate is increasing, (or, equivalently, the velocity of the block is positive,) so it is moving upwards.

(c) At $t = 0.5$, the curve is concave down, so $\left.\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\right|_{t=0.5} < 0$. Its units are cm/s^2. It represents the acceleration of the block at $t = 0.5$ seconds.