1. (20 points, 4 points each) Determine whether the following statements are TRUE or FALSE. **Write out the whole word "TRUE" or "FALSE" for each problem.**

   (a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(2^{\sin(x)}) = \sin(x)2^{\sin(x)-1}\cos(x)$

   (b) The function $\cos(x)$ has all real numbers as domain and $[-1,1]$ as range, so its inverse function $\arccos(x)$ has $[-1,1]$ as domain and all real numbers as range.

   (c) $\dfrac{\mathrm{d}}{\mathrm{d}x}[\cosh^2(\sqrt{x}) - \sinh^2(\sqrt{x})] = 0.$

   (d) $\sin(\arctan(x)) = \dfrac{x}{\sqrt{1-x^2}}.$

   (e) If $f'(x)$ is defined for all x and f has a maximum at $x=1$, then $f'(1)=0$.

Answers: (a) FALSE (b) FALSE (c) TRUE (d) FALSE (e) TRUE

2. (24 points: 8 points each) Find the derivative of the following functions with respect to x.

   (a) $\dfrac{x-6}{x+7}$ (b) $(1+x^2)\arcsin(x)$ (c) $\tan(\ln(1-x))$

   Solutions:

   (a)
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x-6}{x+7}\right) = \frac{(x+7)-(x-6)}{(x+7)^2} = \frac{13}{(x+7)^2}.$$

   (b)
$$\frac{\mathrm{d}}{\mathrm{d}x}\big((1+x^2)\arcsin(x)\big) = 2x\arcsin(x) + (1+x^2)\frac{1}{\sqrt{1-x^2}} = 2x\arcsin(x) + \frac{1+x^2}{\sqrt{1-x^2}}$$

   (c)
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan(\ln(1-x))\right) = \frac{1}{\cos^2(\ln(1-x))}\frac{1}{1-x}(-1) = \frac{-1}{(1-x)\cos^2(\ln(1-x))}$$

3. (9 points) Find the tangent line approximation to $\sqrt{1+x}$ at $x=0$. Use this approximation to estimate $\sqrt{1.02}$.

   Solutions:
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{1+x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left((1+x)^{1/2}\right) = \frac{1}{2}(1+x)^{-1/2} = \frac{1}{2\sqrt{1+x}}$$

   At $x=0$, the slope of the tangent line is $\dfrac{1}{2\sqrt{1+0}} = \dfrac{1}{2}$. The point on the graph with $x=0$ has the corresponding y–coordinate $y = \sqrt{1+x} = \sqrt{1+0} = 1$. Then the tangent line to the graph of $y=\sqrt{1+x}$ at $x=0$ is
$$y - 1 = \frac{1}{2}(x-0) \qquad \text{or equivalently} \qquad y = \frac{1}{2}x + 1.$$

Therefore the tangent line approximation to $\sqrt{1+x}$ at $x = 0$ is

$$\sqrt{1+x} \approx \frac{1}{2}x + 1,$$

and using this approximation we get that

$$\sqrt{1.02} = \sqrt{1+0.02} \approx \frac{1}{2}0.02 + 1 = 1.01.$$

4. (9 points) The part of the graph of

$$\sin(x^2 + y) = x$$

that is near $(0, \pi)$ defines y as a function of x implicitly. Is this function increasing or decreasing near $x = 0$? Explain how you know.

Solutions: Take $\frac{\mathrm{d}}{\mathrm{d}x}$ on both sides of the equation to get

$$\cos(x^2 + y)(2x + \frac{\mathrm{d}y}{\mathrm{d}x}) = 1.$$

Solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$,

$$\begin{aligned} 2x + \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\cos(x^2+y)}, \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\cos(x^2+y)} - 2x. \end{aligned}$$

At the point $(0, \pi)$,

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(0,\pi)} = \frac{1}{\cos(0^2+\pi)} - 2(0) = -1 < 0,$$

so the function is decreasing close to $(0, \pi)$.

5. (7 points) Using the definition of the derivative, show that $\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = -\sin(x)$. You may need to use the following limits:

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1 \qquad \text{and} \qquad \lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta} = 0.$$

Solutions:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\cos(x) &= \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h} \\ &= \lim_{h \to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\ &= \lim_{h \to 0} \frac{\cos(x)(\cosh(h) - 1) - \sin(x)\sin(h)}{h} \\ &= \cos(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} - \sin(x) \lim_{h \to 0} \frac{\sin(h)}{h} \\ &= \cos(x)(0) - \sin(x)(1) = -\sin(x). \end{aligned}$$

6. (9 points: 4, 5)

   (a) Carefully state the Constant Function Theorem.

   (b) Suppose that $f(x)$ is differentiable for all x and that $f'(x) \leq 3$. If $f(0) = 4$, what can you say about the value of $f(2)$? Specify which theorem you are using.

   Solutions:

   (a) Constant Function Theorem: If f is continuous on $[a, b]$ and differentiable on (a, b), and if $f'(x) = 0$ on (a, b), then f is constant on $[a, b]$.

   (b) We may use the Mean Value Theorem, the Decreasing (or Increasing) Function Theorem, or Racetrack Principles to solve this problem. (If you figure out how to use Constant Function Theorem, let me know.)

   Applying the Mean Value Theorem to f on the interval $[0, 2]$, we get that for some c with $0 < c < 2$,
   $$f(2) = f(0) + f'(c)(4 - 2) = 4 + f'(c)(2) \leq 4 + 3(2) = 10.$$

   To use Racetrack Principles, let $h(x) = 3x + 4$. Then $h'(x) = 3$ and $h(0) = 4$. Because $f'(x) \leq h'(x)$ for all x and $f(0) = h(0)$, by Racetrack Principles, $f(x) \leq h(x)$ for all $x \geq 0$. Therefore,
   $$f(2) \leq h(2) = 3(2) + 4 = 10.$$

   We may also apply the Decreasing Function Theorem to the function $f(x) - 3x$. Because $f'(x) - 3 \leq 0$ for all x, it is decreasing. It follows that
   $$f(2) - 3(2) \leq f(0) - 3(0), \quad \text{and so} \quad f(2) \leq f(0) - 0 + 6 = 4 + 6 = 10.$$

7. (22 points total) Consider the function
   $$f(x) = x^4 - 4x^3.$$

   Questions (a)–(f) will help you sketch the graph of $f(x)$.

   (a) (1 point) What is the domain of f?

   (b) (3 points) Is f even, odd, or neither? Why?

   (c) (5 points) Find $f'(x)$. On what interval(s) is f increasing? decreasing? Where are the local max/min points and what are the local max/min values?

   (d) (5 points) Find $f''(x)$. On what interval(s) is f concave up? concave down? Are there any inflection points? If there are, what are their coordinates?

   (e) (3 points) What are the x– and y–intercepts?

   (f) (2 points) What are the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$?

   (g) (3 points) Sketch the graph of f. Make sure that it reflects your answers to all previous parts and mark the points from parts (c), (d) and (e) on your graph.

   Solutions:

   (a) The domain is all real numbers $\mathbb{R}$.

(b) $f(-x) = (-x)^4 - 4(-x)^3 = x^4 + 4x^3 \neq \pm f(x)$, so f is neither even nor odd.

(c)
$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3).$$

$f'(x)$ is positive if $x > 3$ and negative if $x < 3$. Therefore,

$$f \text{ is increasing on } (3, \infty) \text{ and is decreasing on } (-\infty, 3),$$

and f has only a local minimum at $x = 3$. The local minimum value is $f(3) = 3^4 - 4(3^3) = -27$.

(d)
$$f''(x) = 12x^2 - 24x = 12x(x-2).$$

$f''(x)$ is positive if $x < 0$ or $x > 2$ and is negative if $0 < x < 2$. Therefore,

$$f \text{ is concave up on } (-\infty, 0) \text{ and } (2, \infty), \text{ and is concave down on } (0, 2),$$

and f has two inflection points whose coordinates are

$$(0, f(0) = 0), \quad \text{and} \quad (2, f(2) = 2^4 - 4(2^3) = -16).$$

(e) The y–intercept is $f(0) = 0$. The x–intercepts are solutions to the equation

$$0 = f(x) = x^4 - 4x^3 = x^3(x-4),$$

which are $x = 0$ and $x = 4$.

(f) Both limits are $+\infty$ because x^4 is the dominant term.

(g) See picture below.