Math 21: Calculus I **Solutions to Midterm Exam 3** **UC Merced, Fall 2006**

1. Solve the following unrelated problems

   (a) (10 points) Write down, but **DO NOT** evaluate, a Riemann sum to estimate the definite integral $\int_0^1 \sqrt{\tan(x)}\, dx$ using $n = 3$ subintervals and right-endpoints.

   Solutions: Each subinterval has the same length

   $$\Delta x = \frac{1-0}{3} = \frac{1}{3},$$

   and the endpoints of these intervals are

   $$0,\ \frac{1}{3},\ \frac{2}{3},\ 1.$$

   Using left–endpoints 0, $1/3$, and $2/3$, we get that the heights for three rectangles are

   $$\sqrt{\tan(0)},\quad \sqrt{\tan(1/3)},\quad \text{and } \sqrt{\tan(2/3)}.$$

   Hence, the Riemann sum is

   $$\int_0^1 \sqrt{\tan(x)}\, dx \approx \frac{1}{3}\sqrt{\tan(0)} + \frac{1}{3}\sqrt{\tan(1/3)} + \frac{1}{3}\sqrt{\tan(2/3)}.$$

   (b) (10 points) Calculate the exact area enclosed between the x–axis and the parabola $y = 1 - x^2$.

   Solutions: It helps to sketch a graph of the parabola as follows.

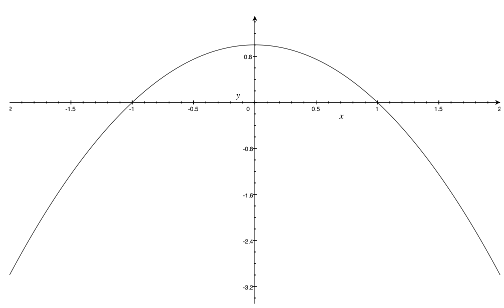

Figure 1: Problem 1(b)

   The paraloba and the x–axis intersect when

   $$0 = 1 - x^2 \Longrightarrow 1 = x^2 \Longrightarrow x = \pm 1.$$

So the area enclosed between the x–axis and the parabola is

$$\int_{-1}^{1}(1-x^2)\,dx = x - \frac{x^3}{3}\Big|_{-1}^{1} = \left[1-\frac{1}{3}\right] - \left[-1-\frac{-1}{3}\right] = \frac{2}{3} - \left[-\frac{2}{3}\right] = \frac{4}{3}.$$

Or, because $1-x^2$ is an even function, a slightly simpler calculation goes as

$$\int_{-1}^{1}(1-x^2)\,dx = 2\int_{0}^{1}(1-x^2)\,dx == 2\left[x-\frac{x^3}{3}\Big|_{0}^{1}\right] = 2\left[1-\frac{1}{3}\right] = \frac{4}{3}.$$

(c) (10 points) Find the limit $\lim_{x\to 0}\left(\frac{1}{\ln(1+x)} - \frac{1}{x}\right)$.

<u>Solutions</u>: First we try to plug in $x=0$ and get that

$$\lim_{x\to 0}\left(\frac{1}{\ln(1+x)} - \frac{1}{x}\right) = \frac{1}{\ln(1+0)} - \frac{1}{0} = \frac{1}{0} - \frac{1}{0} = \infty - \infty.$$

It is an indeterminate form. So we need to rewrite the limit and try l'Hopital's rule.

$$= \lim_{x\to 0}\frac{x-\ln(1+x)}{x\ln(1+x)} = \frac{0-\ln(1+0)}{0\ln(1+0)} = \frac{0}{0}$$

So l'Hopital's rule applies.

$$= \lim_{x\to 0}\frac{1-\frac{1}{1+x}}{\ln(1+x)+x\frac{1}{1+x}} = \frac{1-\frac{1}{1+0}}{\ln(1+0)+0\frac{1}{1+0}} = \frac{1-1}{0+0} = \frac{0}{0}$$

Use l'Hopital's rule a second time.

$$= \lim_{x\to 0}\frac{0+\frac{1}{(1+x)^2}}{\frac{1}{1+x}+\frac{1+x-x}{(1+x)^2}} = \frac{0+1}{1+\frac{1}{1}} = \frac{1}{2}.$$

To summarize,

$$\lim_{x\to 0}\left(\frac{1}{\ln(1+x)} - \frac{1}{x}\right) = \frac{1}{2}.$$

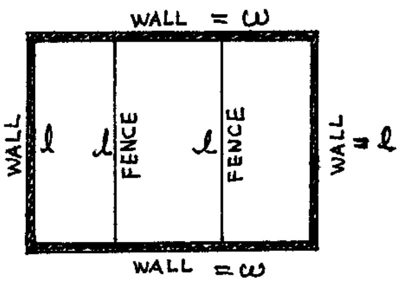

Figure 2: Problem 2

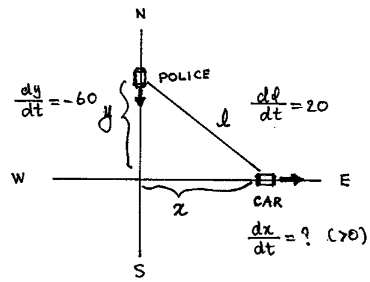

Figure 3: Problem 3

2. (15 points) A field will be made in the shape of a rectangle with an area of 1500 square feet. It is surrounded on all four sides by walls and is divided into three smaller rectangles by fences (see picture). The wall costs \$15 per foot and the fence costs \$10 per foot. What is the lowest possible cost to build such a field?

   Solutions: Call the length of a fence l and the other dimension of the field w, both measured in feet (see picture). Then the cost, in dollars, to build such a field is

$$C = 15(2l + 2w) + 10(2l) = 30l + 30w + 20l = 50l + 30w.$$

   Since the area has to be 1500 square feet,

$$1500 = wl \quad \Longrightarrow \quad w = \frac{1500}{l}.$$

   So the cost C can be expressed as a function of l only

$$C = 50l + 30\frac{1500}{l}.$$

   To find the minimum of C, solve for l from

$$0 = \frac{\mathrm{d}C}{\mathrm{d}l} = 50 - 30\frac{1500}{l^2}.$$

   We get that

$$50 = 30\frac{1500}{l^2} \quad \Longrightarrow \quad 50l^2 = 30 \times 1500 \quad \Longrightarrow \quad l^2 = 30 \times 30 \quad \Longrightarrow \quad l = 30.$$

   The second derivative test

$$\frac{\mathrm{d}^2C}{\mathrm{d}l}\bigg|_{l=30} = 30 \times 2\frac{1500}{l^3}\bigg|_{l=30} > 0$$

shows that $l = 30$ is a minimum point of the cost function C. The minimum cost then is

$$C = 50 \times 30 + 30\frac{1500}{30} = 1500 + 1500 = 3000 \text{ (dollars)}.$$

3. (15 points) A police cruiser, approaching a right–angled intersection from the North, is chasing a speeding car that has turned the corner and is now moving straight East. When the cruiser is 0.6 miles North of the intersection and the car is 0.8 miles to the East, the officer determines with his radar that the distance between himself and the speeding car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

Solutions: Call the distance between the car and the intersection x, that between the police cruiser and the intersection y (see picture). When $x = 0.8$ miles and $y = 0.6$ miles, $\frac{\mathrm{d}y}{\mathrm{d}t} = -60$ mph. If in addition, l denotes the distance between the car and the police cruiser, $\frac{\mathrm{d}l}{\mathrm{d}t} = 20$ mph, and

$$l^2 = x^2 + y^2, \tag{1}$$

from which it follows that $l = \sqrt{x^2 + y^2} = \sqrt{0.8^2 + 0.6^2} = \sqrt{0.64 + 0.36} = \sqrt{1} = 1$. Differentiating equation (1) with respect to t, we get that

$$2l\frac{\mathrm{d}l}{\mathrm{d}t} = 2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} \quad \Longrightarrow \quad l\frac{\mathrm{d}l}{\mathrm{d}t} = x\frac{\mathrm{d}x}{\mathrm{d}t} + y\frac{\mathrm{d}y}{\mathrm{d}t}.$$

Then,

$$1 \times 20 = 0.8\frac{\mathrm{d}x}{\mathrm{d}t} + 0.6 \times (-60) \quad \Longrightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{20 + 0.6 \times 60}{0.8} = \frac{56}{4/5} = \frac{56 \times 5}{4} = 14 \times 5 = 70.$$

The speed of the car is 70 mph.

Only the final answer will be graded for problems 4 and 5. No justification is needed.

4. (20 points, 4 each) Determine whether the following statements are true or false. **Write out the whole word "TRUE" or "FALSE" for each problem.**

   (a) $\displaystyle\int e^x\sqrt{1 + e^x}\,dx = \frac{2}{3}(1 + e^x)^{3/2} + C.$

   (b) $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_1^5 \frac{x^2 + 1}{x}\,dx\right] = 0.$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\int_x^{x^2} \sin(t)\,dt\right] = 2x\sin(x^2) - \sin(x)$.

(d) If $f(x)$ is an even function and $\int_0^1 f(x)\,dx = 2$, then $\int_{-1}^1 f(x)\,dx = 4$.

(e) The definite integral of a function $f(x)$ has the same units as the function $f(x)$ itself.

Solutions:

(a) True. (Take derivative of the right hand side using the chain rule, we get the integrand of the left hand side, and there is a constant C as well.)

(b) True. (The value of a definite integral is a number and the derivative of a number is 0.)

(c) True. (Straightforward calculation of derivatives using the Fundamental Theorem of Calculus, properties of integrals and the chain rule.)

(d) True. (The graph of an even function is symmetric about the y–axis.)

(e) False. (The units of a definite integral of $f(x)$ are those of $f(x)$ times those of x.)

5. (20 points, 4 each) The graph of the **derivative of** $\mathbf{F}(\mathbf{x})$ is given below. Determine whether the following statements about $\mathbf{F}(\mathbf{x})$ are true or false. **Write out the whole word "TRUE" or "FALSE" for each problem.**

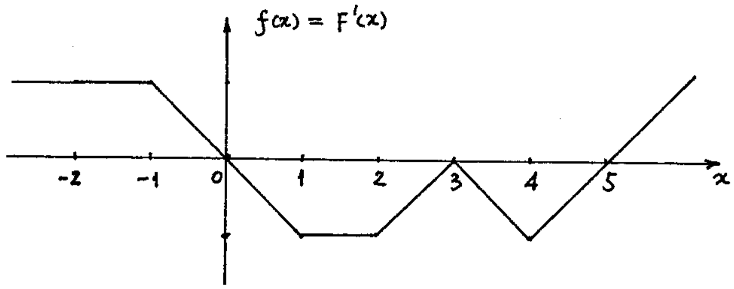

Figure 4: Problem 5

(a) $F(x)$ is constant between $x = -2$ and $x = -1$.

(b) $F(x)$ has a local maximum at $x = 5$.

(c) $F(x)$ is concave down between $x = -1$ and $x = 1$.

(d) $F(x)$ has inflection points at $x = 3$ and $x = 4$.

(e) $F(2) = F(-2)$.

<u>Solutions</u>:

(a) False. ($F(x)$ is increasing between $x = -2$ and $x = -1$.)

(b) False. ($F(x)$ has a local minimum at $x = 5$.)

(c) True. (Because $F'(x)$ is decreasing between $x = -1$ and $x = 1$.)

(d) True. (Inflection points are where $F(x)$ changes concavity.)

(e) True. (The Fundamental Theorem of Calculus implies that $F(2) = F(-2) + \int_{-2}^{2} f(x)\,dx = F(-2) + 0$.)