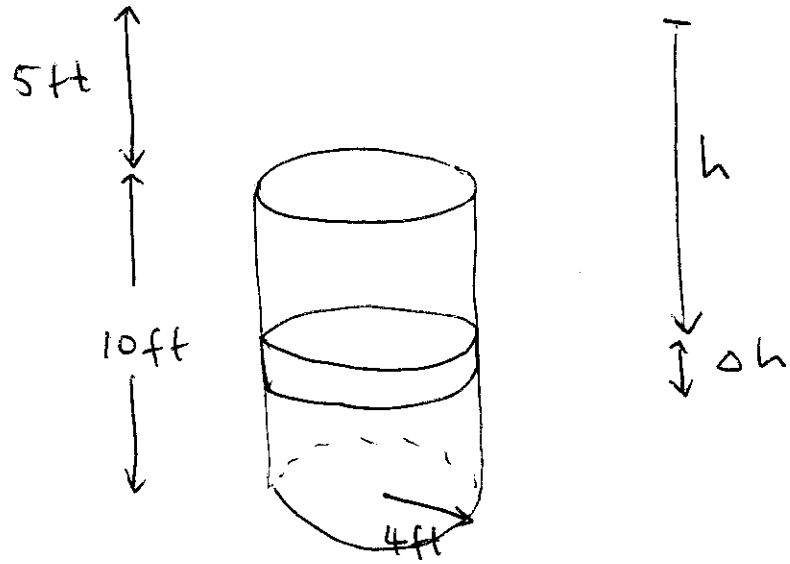
Math 22 Unit 3 Exam Solutions.

1.



Weight of slice $= \pi \cdot 4^2 \cdot \Delta h \cdot 62.4$ lb
$= 998.4\pi \, \Delta h$ lb

Work done on slice $\approx 998.4\pi \, \Delta h \cdot h$
$= 998.4\pi h \, \Delta h$ ft-lb

Total work done $= \lim_{\Delta h \to 0} \sum 998.4\pi h \, \Delta h$

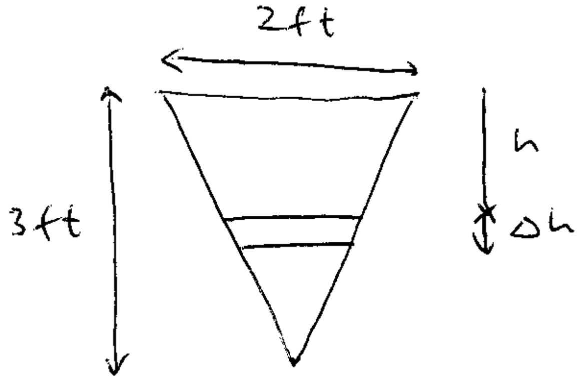
$$= \int_5^{15} 998.4\pi h \, dh$$

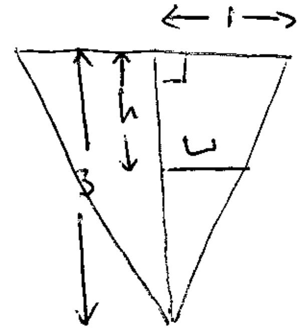
$$= 998.4\pi \left[\tfrac{1}{2} h^2 \right]_5^{15}$$

$$= 499.2\pi \left(15^2 - 5^2 \right)$$

$$= 99840\pi \text{ ft-lb.}$$

2.





$$\frac{1}{3} = \frac{w}{3-h} \quad \text{or} \quad w = \frac{1}{3}(3-h)$$

$$\text{Area of slice} \simeq 2w\,\Delta h = \frac{2}{3}(3-h)\,\Delta h. \ \text{ft}^2$$

$$\text{Force on slice} \simeq 62.4h \cdot \frac{2}{3}(3-h)\,\Delta h \quad \text{lb}$$

$$= 41.6\,h(3-h)\,\Delta h \quad \text{lb}.$$

$$\text{Total force} = \lim_{\Delta h \to 0} \sum 41.6\,h(3-h)\,\Delta h$$

$$= \int_0^3 41.6\,h(3-h)\,dh$$

$$= 41.6\left[\frac{3}{2}h^2 - \frac{1}{3}h^3\right]_0^3$$

$$= 41.6\left(\frac{3}{2}\cdot 3^2 - \frac{1}{3}\cdot 3^3\right)$$

$$= 187.2 \ \text{lb}.$$

3. Let $f(x) = \dfrac{\ln x}{x}$.

Now $f'(x) = \dfrac{\frac{1}{x} \cdot x - 1 \cdot \ln x}{x^2}$

$$= \frac{1 - \ln x}{x^2} < 0 \quad \text{if} \quad x > e$$

Hence, $f(x)$ is decreasing and positive for $x > e$. Now,

$$\int_e^\infty f(x)\,dx = \int_e^\infty \frac{\ln x}{x}\,dx$$

$$= \lim_{b \to \infty} \int_e^b \frac{\ln x}{x}\,dx \qquad u = \ln x$$

$$= \lim_{b \to \infty} \frac{1}{2} u^2 \Big|_1^{\ln b}$$

$$= \lim_{b \to \infty} \frac{1}{2} (\ln b)^2 - \frac{1}{2}$$

$$= \infty$$

Hence $\int_e^\infty f(x)\,dx$ diverges and so therefore does the series.

4. We have that

$$n^3 - 5 < n^3$$

$$\Rightarrow \frac{1}{n^3 - 5} > \frac{1}{n^3} \qquad \text{for } n \geq 2$$

$$\Rightarrow \frac{n^2}{n^3 - 5} > \frac{n^2}{n^3} = \frac{1}{n}. \qquad \text{for } n \geq 2$$

Now $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series) and so by the comparison test $\sum_{n=1}^{\infty} \frac{n^2}{n^3-5}$ diverges.

5. Let $a_n = \frac{1}{\sqrt{n}}$. Now,

$$n < n+1$$

$$\Rightarrow \quad \sqrt{n} < \sqrt{n+1}$$

$$\Rightarrow \quad \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$$

$$\Rightarrow \quad a_n > a_{n+1}$$

Also, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Hence, the series converges.

6. Let $a_n = \frac{n^3(x-1)^n}{7^{n+1}}$.

Now,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^3(x-1)^{n+1}}{7^{(n+1)+1}}}{\frac{n^3(x-1)^n}{7^{n+1}}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^3(x-1)^{n+1}}{7^{n+2}} \cdot \frac{7^{n+1}}{n^3(x-1)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^3(x-1)}{7n^3} \right|$$

$$= \frac{1}{7}|x-1| \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^3$$

$$= \frac{1}{7}|x-1|.$$

By the ratio test, the series converges if

$$\frac{1}{7}|x-1| < 1$$

That is, if $\quad |x-1| < 7$

Hence $R = 7$.

When $x = 8$, $\displaystyle\sum_{n=1}^{\infty} \frac{n^3(8-1)^n}{7^{n+1}} = \sum_{n=1}^{\infty} \frac{n^3 \cdot 7^n}{7^{n+1}}$

$$= \sum_{n=1}^{\infty} \frac{n^3}{7}$$

Now, $\displaystyle\lim_{n\to\infty} \frac{n^3}{7} = \infty \neq 0$ and so the series diverges.

When $x = -6$, $\displaystyle\sum_{n=1}^{\infty} \frac{n^3(-6-1)^n}{7^{n+1}} = \sum_{n=1}^{\infty} \frac{n^3(-7)^n}{7^{n+1}}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n n^3}{7}$$

Now, $\displaystyle\lim_{n\to\infty} \frac{(-1)^n n^3}{7} \neq 0$ and so again the series diverges. Hence the interval of convergence is $-6 < x < 8$.

7. $f(x) = \sin(2x)$; $f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{2} = 1$

$f'(x) = 2\cos(2x)$; $f'\left(\frac{\pi}{4}\right) = 2\cos\frac{\pi}{2} = 0$

$f''(x) = -4\sin(2x)$; $f''\left(\frac{\pi}{4}\right) = -4\sin\frac{\pi}{2} = -4$

$f'''(x) = -8\cos(2x)$; $f'''\left(\frac{\pi}{4}\right) = -8\cos\frac{\pi}{2} = \cancel{8}\ 0$

$f^{(4)}(x) = 16\sin(2x)$; $f^{(4)}\left(\frac{\pi}{4}\right) = 16\sin\frac{\pi}{2} = 16$

Hence, the degree 4 Taylor polynomial about $x = \frac{\pi}{4}$ is

$$f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(x-\frac{\pi}{4}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{4}\right)}{4!}\left(x-\frac{\pi}{4}\right)^4$$

$$= 1 + 0\cdot\left(x-\frac{\pi}{4}\right) + \frac{(-4)}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{0}{3!}\cdot\left(x-\frac{\pi}{4}\right)^3 + \frac{16}{4!}\left(x-\frac{\pi}{4}\right)^4$$

$$= 1 - 2\left(x-\frac{\pi}{4}\right)^2 + \frac{2}{3}\left(x-\frac{\pi}{4}\right)^4.$$