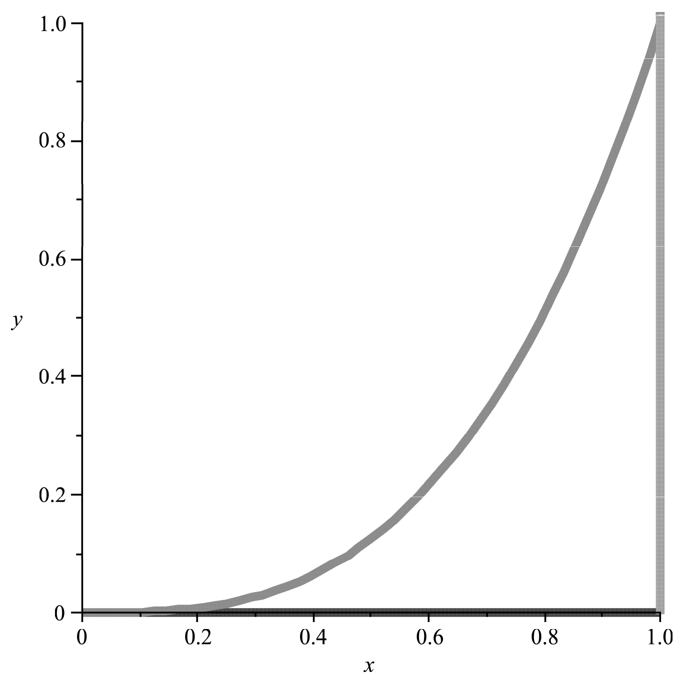
1. A 10 meter high monument has circular horizontal cross sections each with area $A(x) = 10\exp(-x^2/100)$,where x is the height in meters from the ground. Write out, but do not evaluate, an integral representing the volume of the monument.

   Solution:
$$\int_0^{10} 10\exp(-x^2/100).$$

For Problems #2 and #3, find the volume of the described solid of revolution.

2. The region bounded by the curves $x = 1$, $y = 0$ and $y = x^3$ is rotated around the x-axis.

   Solution: The region in question lies within the thickened boundaries in the following picture:



This region is rotated around the y-axis, so there are two ways of computing the volume.

(a) By washers:

$$\int_0^1 \pi(1 - y^{\frac{2}{3}})dy,$$

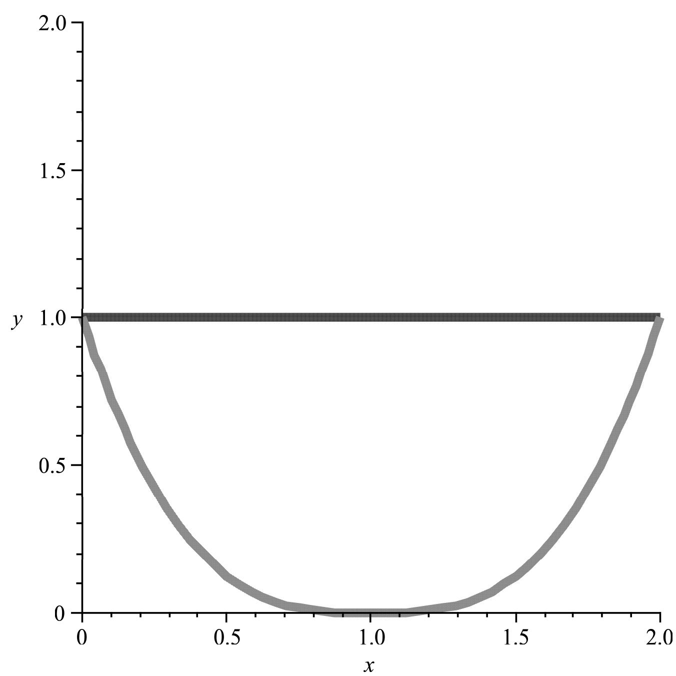
(b) or by cylindrical shells:

$$\int_0^1 2\pi x \cdot x^3 dx.$$

Both integrals give the value $2\pi/5$.

3. The region bounded by the curves $y = 1$, $y = |x - 1|^3$ is rotated around the x-axis.

Solution: The region is that bounded by the following thickened curves:



Again, we have two methods to find the resulting volume.

(a) The washer method:

$$\int_0^2 \pi(1-|x-1|^6)dx,$$

(b) or the cylindrical shell method:

$$\int_0^1 2\pi y \cdot 2y^{\frac{1}{3}} dy.$$

The second expression is due to the fact that the boundaries with respect to the y-coordinate are the curves $x = 1 + y^{\frac{1}{3}}$ and $x = 1 - y^{\frac{1}{3}}$, and the difference between the varying x values, each of which constitutes the "height" of the cylinder, is given by $2y^{\frac{1}{3}}$.

We evaluate the first integral by noting that since $|x-1|$ is being raised to the 6th power, we can drop the absolute value signs (why?). The substitution $u = x - 1$, $du = dx$ then yields

$$\int_{-1}^1 \pi(1-u^6)du.$$

By FTC, this can be evaluated as

$$\pi u - \pi \frac{u^7}{7}\Big|_{-1}^{1} = 2\pi - \frac{2}{7}\pi = \frac{12\pi}{7}.$$

We can also use the second integral above and get

$$\int_0^1 4\pi y^{\frac{4}{3}} dy = 4\pi \frac{3}{7} y^{\frac{7}{3}}\Big|_0^1 = \frac{12\pi}{7}.$$

4. A cylindrical water tank with radius 3 meters and height 10 meters is supported by metal stilts so that its base is 10 meters from the ground. Water at ground level is pumped up into the tank until it is full. How much energy does this require? Recall that water weighs 9800 newtons per cubic meter. State your answer in joules (= newton-meters).

   Solution: For $10 \leq x \leq 20$, there is a layer of water with volume Area of cross section $\triangle x$. The area of a cross section is given by $\pi 3^2 = 9\pi$. With the information about the weight per unit volume of water, the weight of a cross is $9800 \cdot 9\pi \triangle x = 88,200\pi \triangle x$. The amount of work required to lift a cross section to height x is given by $88,200\pi x \triangle x$. Thus the total work required is given by the following integral.

$$\int_{10}^{20} 88,200\pi x dx.$$

   Evaluating, we get

$$44,100\pi x^2\Big|_{10}^{20} = 13,230,000\pi.$$

5. Find the average value of the function $f(x) = \sqrt{x}$ on the interval $[0, 100]$. For which value x in the interval does $f(x)$ take the average value?

Solution: The average value is given by the expression:

$$\frac{1}{100-0}\int_0^{100} \sqrt{x}dx = \frac{1}{100}\,\frac{2}{3}x^{\frac{3}{2}}\Big|_0^{100} = \frac{20}{3}.$$

The function takes this value at x_0 such that $\sqrt{x_0} = \frac{20}{3}$, i. e. $x_0 = \frac{400}{9}$.