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Math 22 Spring 2008

Math 22 Midterm 2: Spring 2008 *Solutions*

1. (15 points) Write the correct form of the partial fraction decomposition of the rational function below. Find the numerator of the terms whose denominator is $(x+1)^2$

$$\frac{x^2+x+3}{(x+1)^2(x^2+5)}$$

Solution:

1. We need the form of the decomposition, in other words, the setup before finding the constants in any partial fractions situation.

$$\frac{x^2+x+3}{(x+1)^2(x^2+5)} = \boxed{\frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{Cx+D}{(x^2+5)}}$$

Afterwards, you are asked to find the numerator of the term whose denominator is $(x+1)^2$, in other words, *B*.

To do so, we need to set a common denominator of $\underline{(x+1)^2\ (x^2+5)}$, which will be seen below in order to equate the numerators:

$$x^2+x+3 = A(x+1)(x^2+5) + B(x^2+5) + (Cx+D)(x+1)^2$$

Now, we let *x* = -1, so the terms with *A, C,* and *D*, cancel out (in other words, or go to zero)

$\rightarrow 3 = 6B$ $\boxed{B = 1/2}$

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2. (15 points) Determine which of the following expressions is correct if one uses Simpson's Rule to approximate the integral $\int_0^3 f(x)\,dx$ with *n* = 4.

(a) $\frac{1}{4}\left[f\left(\frac{3}{8}\right)+4f\left(\frac{9}{8}\right)+2f\left(\frac{15}{8}\right)+4f\left(\frac{21}{8}\right)\right]$

(b) $\frac{1}{4}\left[f(0)+4f\left(\frac{3}{4}\right)+2f\left(\frac{3}{2}\right)+4f\left(\frac{9}{4}\right)+f(3)\right]$

(c) $\frac{3}{8}\left[f(0)+2f\left(\frac{3}{4}\right)+2f\left(\frac{3}{2}\right)+2f\left(\frac{9}{4}\right)+f(3)\right]$

(d) $\frac{3}{4}\left[f(0)+f\left(\frac{3}{4}\right)+f\left(\frac{3}{2}\right)+f\left(\frac{9}{4}\right)\right]$

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Solution:
This multiple choice question can be done either by using process of elimination or knowledge of the formula for Simpson's Rule.

First, we know our bounds range from 0 to 3. So our *(b-a)* =3

So our $\Delta x = \left(\frac{b-a}{n}\right) \rightarrow \frac{3}{4}$ but since Simpson's Rule requires our Δx to be divided by 3,

$\Delta x_{Simpson's\ Rule} = \frac{3/4}{3} = \frac{1}{4}$ This immediately eliminates choices C and D.

From there, the basic approximation start point starts from *f (a)= f (0)* for the Simpson's Rule*,* rather than *f(3/8)*, which eliminates choice A, so with all of the other choices eliminated,

$\therefore$ Choice B is correct

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3. (15 points) Determine whether the following improper integral converges. If it does converge, compute the value of the integral.

$$\int_0^\infty x^2 e^{-x}\, dx$$

Solution:

3. We need to use integration by parts for the integral: $\int_0^\infty x^2 e^{-x} dx$

$u = x^2$
$du = 2x\, dx$
$v = -e^{-x}$
$dv = e^{-x} dx$

$\rightarrow \frac{x^2}{e^x} + \int 2xe^{-x}\, dx$ Now we use integration by parts again.

$u = 2x$
$du = 2\, dx$
$v = -e^{-x}$
$dv = e^{-x}\, dx$

$\rightarrow \frac{x^2}{e^x} - 2xe^{-x} + \int 2e^{-x}\, dx$ $\therefore \left[x^2 e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^\infty$

Applying the first bound of infinity requires the use of L'Hospital's Rule on the first term twice and second term once, but the whole part yields 0 when infinity is applied.

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When 0, is applied, both of the first two terms go to 0, but the last one goes to -2 in the following way:

$$\Rightarrow (0-0-0)-\left(0-0-\frac{2}{e^0}\right)= 2$$

$\therefore$ The integral is **convergent** with a limit of 2.

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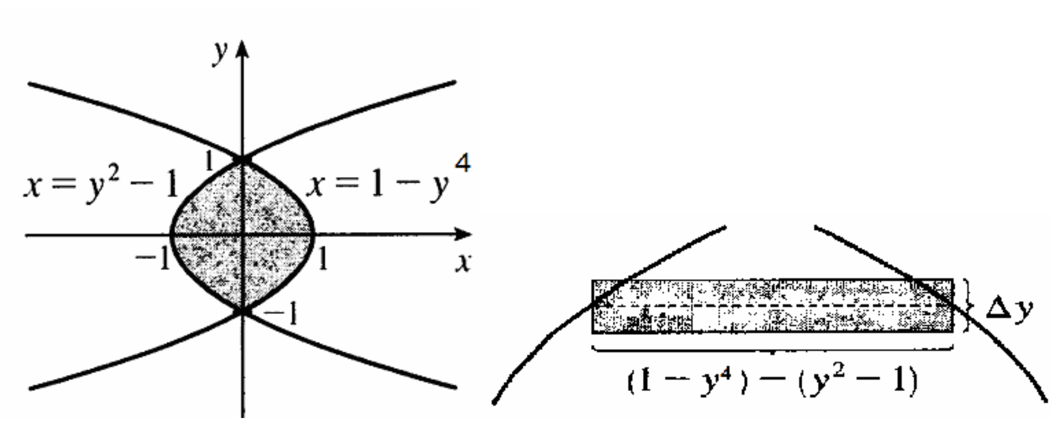
4. (15 points) Sketch the finite region bounded by the two curves below. Find the area of this region.

$$x = y^2 - 1,\ x = 1 - y^4$$

Solution:

This is what the region looks like along with its viewing rectangle:
Now, we integrate with respect to y to find the area by subtracting the left side of the region from the right as shown in the rectangle.



$$A = \int_{-1}^{1}\left[(1-y^4)-(y^2-1)\right]dy$$

$$\Rightarrow A = \int_{-1}^{1}\left[(1-y^4-y^2+1\right]dy$$

$$\Rightarrow A = \int_{-1}^{1}\left[(1-y^4-y^2+1\right]dy \Rightarrow \left[2y-\frac{y^5}{5}-\frac{y^3}{3}\right]_{-1}^{1} \Rightarrow (2-\frac{1}{5}-\frac{1}{3})-(-2+\frac{1}{5}+\frac{1}{3})$$

$$\Rightarrow (\frac{30}{15}-\frac{3}{15}-\frac{5}{15})+(\frac{30}{15}-\frac{3}{15}-\frac{5}{15}) \Rightarrow \boxed{\frac{44}{15}}$$

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5. (15 points) The integral $\int_0^{\pi} \pi \sin^2 x \, dx$ can be interpreted as the volume of a solid of revolution. Using a geometric interpretation, find the integral in the list below which yields the same value. Note that it is possible to do this problem without computing any integral.

(a) $2\int_0^1 2\pi y\left(\frac{\pi}{2} - \sin^{-1} y\right) dy$

(b) $\int_0^{\pi} 2\pi x \sin x \, dx$

(c) $\int_0^{\pi} \sin x \, dx$

(d) $\int_0^1 \pi \left[(\pi - \sin^{-1} y)^2 - (\sin^{-1} y)^2\right] dy$

Solution:

We are given that $\int_0^{\pi} \pi \sin^2 x \, dx$ is an expression of volume of revolution.

However, this is just a simple expression for the disc method of rotating the curve of $f(x) = \sin x$ ($0 \le x \le \pi$) about the *x*-axis.

We can use the method of cylindrical shells to do evaluation the same volume of revolution as we were given.

Choice C seems to just be the method of finding the area under the curve of $f(x) = \sin x$ ($0 \le x \le \pi$). We needed a volume expression so that is eliminated.

For the method of cylindrical shells, we use the corresponding *y*-values of the graph of $f(x) = \sin x$. Choice B is incorrect because of both the wrong bounds.

Choice A shows that the bounds for rotation for normal *xy* rotation about the *x-axi*s are 0 to $\pi / 2$. However, they have doubled the resulting rotation with the correct radius. With shells, rotation about the *x*-axis uses *y* bounds.

Therefore, Choice A is correct.

6. (15 points) Suppose the curve *C* can be represented in two ways: $y = f(x), a \le x \le b$; or $x = g(y), c \le y \le d$. Let *S* be the surface of revolution obtained by rotating *C* around the *x*-axis. Which *two* of the integrals below give the surface area of *S*?

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(a) $\int_c^d 2\pi\, g(y)\sqrt{1+g'(\mathrm{y})^2}\,dy$

(b) $\int_a^b 2\pi\, x\sqrt{1+f'(\mathrm{x})^2}\,dx$

(c) $\int_a^b 2\pi\, f(x)\sqrt{1+f'(\mathrm{x})^2}\,dx$

(d) $\int_c^d 2\pi\, y\sqrt{1+g'(\mathrm{y})^2}\,dy$

Solution:
This can also be done by either process of elimination or knowledge of the manipulations of the surface area of revolution formulas. We can eliminate Choice B because the expression $2\pi x$ is used for rotation about the *y*-axis. In addition, Choice A is incorrect because that is also rotation about the *x*-axis. Choice C is correct because of the correct bounds and notation for the specified axis of rotation. With the other choices either chosen or eliminated, Choice D is the remaining correct one because is also in the format of Choice C in simplified Leibniz notation.

$\therefore$ $\boxed{\text{Choices C and D}}$

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7. (10 points) Write out, but do not evaluate, the integral whose value is the length of the curve $x = e^{-y^2}$, $0 \le y \le 1$.

Solution:
For this problem, we can use the formula for arc length in terms of *y*, which is

$L = \int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ Since our bounds for *y* in terms of *x* are known (0 and 1), we need to just differentiate $x = e^{-y^2}$ with respect to *y* by using the Chain Rule and then square our resulting derivative.

Thus, $\frac{dx}{dy} = -2ye^{-y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = 4y^2e^{-2y^2}$

From there, we just substitute with our solved information:

$$\boxed{\Rightarrow L = \int_0^1 \sqrt{1+4y^2e^{-2y^2}}\,dy}$$