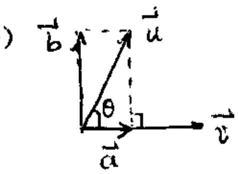
Math 23, Spring 2008

Solutions to Midterm 1

1. (a) a normal vector $= \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ 0 & 1 & 2 \end{vmatrix} = -7\vec{i} - 2\vec{j} + \vec{k}$

$\Rightarrow$ the plane is $\boxed{-7(x-2) - 2(y-5) + (z-3) = 0}$ or

$\boxed{-7x - 2y + z + 21 = 0}$

(b) 

$\vec{a} = \text{proj}_{\vec{v}}\, \vec{u} = |\vec{u}| \cos\theta \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|} \dfrac{\vec{v}}{|\vec{v}|}$

$\vec{u}\cdot\vec{v} = 0 - 2 + 6 = 4$, $|\vec{v}|^2 = 1^2 + 2^2 = 5$

$\Rightarrow \boxed{\vec{a} = \frac{4}{5}\vec{v} = \frac{4}{5}\langle 0, 1, 2\rangle}$

$\vec{b} = \vec{u} - \vec{a} = \langle 1, -2, 3\rangle - \frac{4}{5}\langle 0, 1, 2\rangle = \langle 1, -2 - \frac{4}{5}, 3 - \frac{8}{5}\rangle$

$\boxed{\vec{b} = \langle 1, -\frac{14}{5}, \frac{7}{5}\rangle}$

2. (a) $\boxed{\begin{aligned} x &= 3\cos t \\ y &= 3\sin t \\ z &= 1 - y = 1 - 3\sin t \end{aligned}}$

(b) $\vec{r}\,'(t) = \langle 3\cos 3t, 4, -3\sin 3t\rangle$, $|\vec{r}(t)| = \sqrt{9\cos^2(3t) + 16 + 9\sin^2(3t)}$

$= \sqrt{9 + 16} = 5$

$L = \int_0^2 |\vec{r}\,'(t)|\,dt = \int_0^2 5\,dt = 10$

(c) $(0, 0, 1) \longleftrightarrow t = 0$. tangent vector $= \vec{r}\,'(0) = \langle 3, 4, 0\rangle$

$\Rightarrow$ tangent line $\boxed{x = 3t, \quad y = 4t, \quad z = 1}$

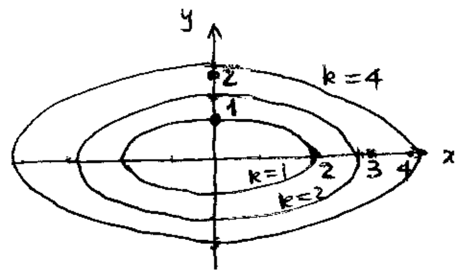
3. (a) $k = \sqrt{x^2 + 4y^2 - 4} \Rightarrow k^2 = x^2 + 4y^2 - 4 \Rightarrow x^2 + 4y^2 = k^2 + 4$

$\Rightarrow \dfrac{x^2}{4} + y^2 = \dfrac{k^2}{4} + 1$, $k \geq 0$ — ellipses as level curves

$k=0 \qquad \frac{x^2}{4} + y^2 = 1$

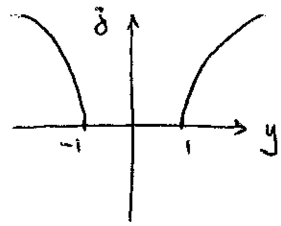
$k=2 \qquad \frac{x^2}{4} + y^2 = 2$

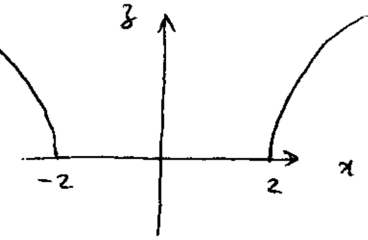
$k=4 \qquad \frac{x^2}{4} + y^2 = 5$



(b) $x=0 \Rightarrow z = f(0,y) = \sqrt{4y^2-4} \Rightarrow z^2 = 4y^2-4, \; z \geq 0$

$\Rightarrow 4y^2 - z^2 = 4, \; z \geq 0 \Rightarrow y^2 - \frac{z^2}{4} = 1, \; z \geq 0$ — part of hyperbola

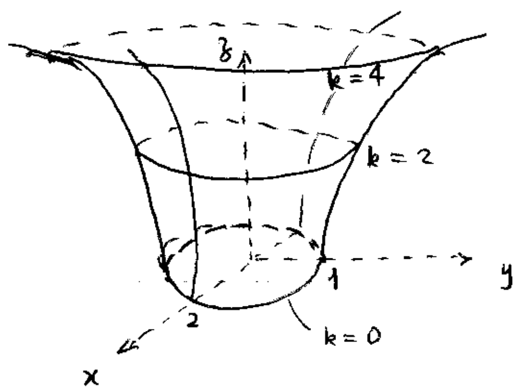




$y=0 \Rightarrow z = f(x,0) = \sqrt{x^2-4} \Rightarrow z^2 = x^2-4, \; z \geq 0$

$\Rightarrow x^2 - z^2 = 4, \; z \geq 0$ — part of hyperbola

(c)



— upper-half of a hyperboloid of one sheet

(d) $f_x = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+4y^2-4}} = \frac{x}{\sqrt{x^2+4y^2-4}}, \quad f_y = \frac{\overset{4}{\cancel{8}}y}{\cancel{2}\sqrt{x^2+4y^2-4}} = \frac{4y}{\sqrt{x^2+4y^2-4}}$

$f_x(1,1) = \frac{1}{\sqrt{1+4-4}} = 1, \quad f_y(1,1) = \frac{4}{\sqrt{1+4-4}} = 4$

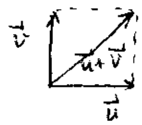
4 (a) $s(t) = \int_0^t |\vec{r}'(u)|\,du \Rightarrow \frac{ds}{dt} = |\vec{r}'(t)| = \left|\frac{d\vec{r}}{dt}\right| = 1 \Rightarrow t$ is arc length.

(b) No. For example, $\vec{u} \times \vec{u} = 0$ for any $\vec{u}$, even if $\vec{u} \neq \vec{0}$.

(c) If we get different limits when we let $(x,y) \to (0,0)$

along different paths, then $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(d) $f(x,y) = \begin{cases} 1, & xy = 0 \\ 0, & xy \neq 0 \end{cases}$ $\quad (a,b) = (0,0)$

(e) 

$|\vec{u}| = |\vec{v}| = 1 \Rightarrow |\vec{u}+\vec{v}| = \sqrt{2}$