Math 23, spring 08 Solutions to Midterm 2

1. (a) $f(0,2) = 5-3(0)+4(2) = 5+8 = \boxed{13}$ because the tangent plane to $y=f(x,y)$ at $(0,2)$ goes thru the same point $(0,2,f(0,2))$ as the graph $y=f(x,y)$ does.

(b) $f(x,y)$ increases the fastest in the direction of $\vec{\nabla} f(0,2)$ at $(0,2)$.

$f_x(0,2) = -3,\ f_y(0,2) = 4 \Rightarrow \vec{\nabla} f(0,2) = \boxed{\langle -3,4 \rangle}$.

(c) $D_{\vec{v}} f(0,2) = \vec{\nabla} f(0,2) \cdot \frac{\vec{v}}{\|\vec{v}\|} = \langle -3,4\rangle \cdot \frac{\langle -1,-1\rangle}{\sqrt{2}} = \frac{3-4}{\sqrt{2}} = \boxed{-\frac{1}{\sqrt{2}}}$

(d) $t=0 \Rightarrow x = \sin(0) = 0,\ y = 2e^0 = 2.$

$$\frac{df}{dt}\Big|_{t=0} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\Big|_{t=0} = -3\cos t + 4 \cdot 2e^t \Big|_{t=0} = -3 \cdot 1 + 8e^0 = \boxed{5}$$

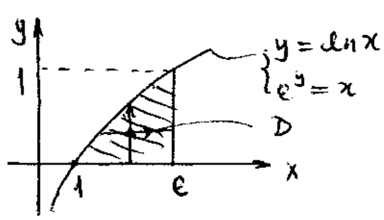
2. (a) $\begin{cases} 0 = f_x = 2x-2 \Rightarrow x=1 \\ 0 = f_y = 2y \Rightarrow y=0 \end{cases}$ critical pt $\boxed{(1,0)}$

At $(0,0)$ $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0,\quad f_{xx} = 2 > 0 \Rightarrow (1,0)$ is a $\boxed{\text{local min}}$

(b) on the boundary $x^2+y^2=4$, $f(x,y) = x^2-2x+(4-x^2) = -2x+4$.

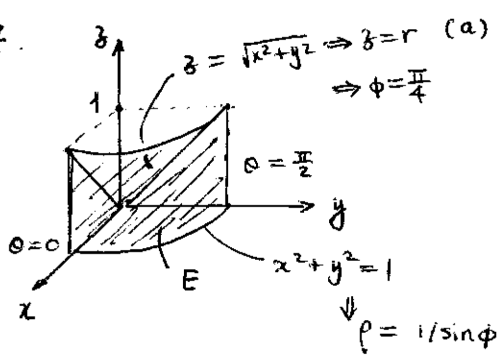
$$\left.\begin{array}{l} -2x+4 \text{ has a min. when } x=2,\ y=0 :\ -2(2)+4 = 0 \\ -2x+4 \text{ " " max " } x=-2,\ y=0 :\ -2(-2)+4 = 8 \\ f(1,0) = 1^2 - 2(1) + 0^2 = -1 \end{array}\right\} \Rightarrow \boxed{\begin{array}{l}\text{abs. max of } f \text{ is } 8 \\ \text{abs. min of } f \text{ is } -1\end{array}}$$

3.



$$\text{Area}(D) = \boxed{\int_1^e \int_0^{\ln x} 1\, dy\, dx} = \boxed{\int_0^1 \int_{e^y}^{e} 1\, dx\, dy}$$

4.



$$\text{mass} = \iiint_E d(x,y,z)\, dV = \boxed{\int_0^{\pi/2} \int_0^1 \int_0^r z\, r\, dz\, dr\, d\theta}$$

$x = r\cos\theta,\quad y = r\sin\theta,\quad z = z$

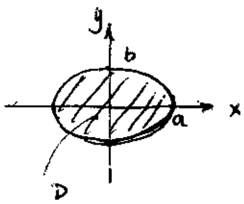
(b) $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$

$$\text{mass} = \boxed{\int_0^{\pi/2} \int_{\pi/4}^{\pi/2} \int_0^{1/\sin\phi} \rho\cos\phi \; \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$

cone : $z = \sqrt{x^2+y^2} \Rightarrow \rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta} = \rho\sin\phi \Rightarrow \cos\phi = \sin\phi \Rightarrow \phi = \frac{\pi}{4}$

cylinder: $x^2+y^2=1 \Rightarrow \rho\sin\phi = 1 \Rightarrow \rho = 1/\sin\phi$.

5.



$$\begin{cases} x = au \\ y = bv \end{cases} \Rightarrow \frac{(au)^2}{a^2} + \frac{(bv)^2}{b^2} = 1 \Rightarrow u^2 + v^2 = 1$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

$$\text{Area} = \iint_D 1 \, dA = \iint_{\{u^2+v^2 \le 1\}} 1 \, (ab) \, du dv = (ab)(\text{area of a unit circle})$$

$$= (ab)\,\pi(1^2) = \pi ab .$$