UC Merced: MATH 24 — Exam #1 — 28 September 2006

On the front of your bluebook print (1) your name, (2) your student ID number, (3) your instructor's name (Sprague) and (4) a grading table. Show all work in your bluebook and BOX IN YOUR FINAL ANSWERS where appropriate. A correct answer with no supporting work may receive no credit while an incorrect answer with some correct work may receive partial credit. Textbooks, class notes, and calculators are not permitted. You are permitted on 8.5×11 inch crib sheet. There are a total of five problems and a total of 100 points. Please start each of the five problems on a new page.

1. **(20 points)** Answer the following Always True (T) or False (F). Only your final answer will be graded on these problems.

   (a) The advantage of the variation-of-parameters approach for linear first-order ODEs is that a solution can always be found that is considered "closed-form."

   (b) $y(t) = t\cos(t)$ is a solution of the IVP $\dfrac{dy}{dt} - \dfrac{y}{t} = -t\sin(t), \quad y(0) = 0.$

   (c) Picard's theorem tells us that the solution to the IVP $\dfrac{dy}{dt} = 2\sqrt{y}, y(0) = 0$, is not unique, because $\frac{\partial}{\partial y}\left(2\sqrt{y}\right)$ is undefined at $y = 0$.

   (d) If $y_1(t)$ and $y_2(t)$ are solutions to the differential equation $\dfrac{d^2y}{dt^2} + \cos(t)t^2\dfrac{dy}{dt} = \ln(t)$, then $y_3(t) = y_1(t) + y_2(t)$ is also a solution.

2. **(45 points total)**

   (a) **(10 points)** Using the classifications discussed in this course to date, classify the following differential equations for $y(t)$ as thoroughly as you can.

   $$(1)\ t\frac{dy}{dt}\frac{dy}{dt} + y = \cos(t) \qquad (2)\ \frac{d^2y}{dt^2} + ty = y$$

   (b) **(15 points)** Calculate the solution to the following IVP, and simplify as much as possible. Assuming that you can find a solution, what can you say about the uniqueness of your solution?

   $$\frac{dy}{dt} = \frac{1+y^2}{2y}\frac{1}{t}, \quad y(1) = 1.$$

   (c) **(10 points)** What would the integrating factor be, if you were to use the integrating-factor method to solve the following for $0 \le t \le \pi/2$. Simplify your answer as much as possible.

   $$\cos(t)\frac{dy}{dt} = [y + \cos(t)]\sin(t)$$

   (d) **(5 points)** Sketch the phase line for the following autonomous DE. Choose your range to include any equilibrium solutions, which should be clearly indicated as stable, unstable, or semi-stable.

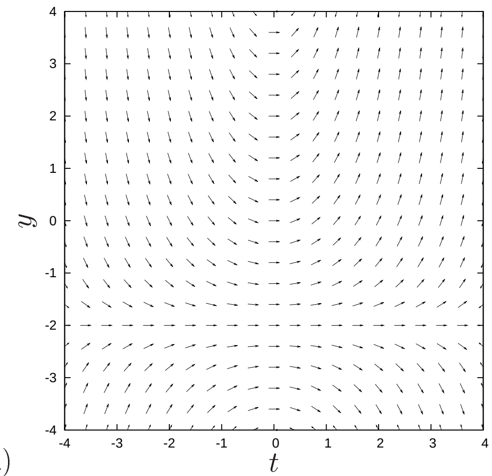
   $$\frac{dy}{dt} = 2(y-1)y$$

   (e) **(5 points)** Consider the Bernoulli equation $\dfrac{dy}{dt} - t\tan(t)y = y^3$. Show that the transformation $y = u^{-1/2}$ reduces the equation to a linear equation in u. You do not need to solve the equation.

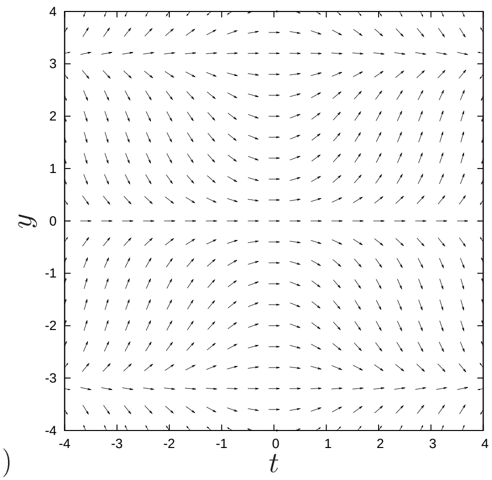
3. **(5 points)** Write down a mathematical model for the following statement: In some forest, the change in the number of rabbits at a given time is proportional to the square root of the number of rabbits in the forest at that time.

4. **(20 points)** For each of the following four differential equations, identify the appropriate direction field (A) – (D). For equation (3) only, explicitly identify any equilibrium solutions within the range shown, and qualify any such solutions as stable, unstable, or neither.

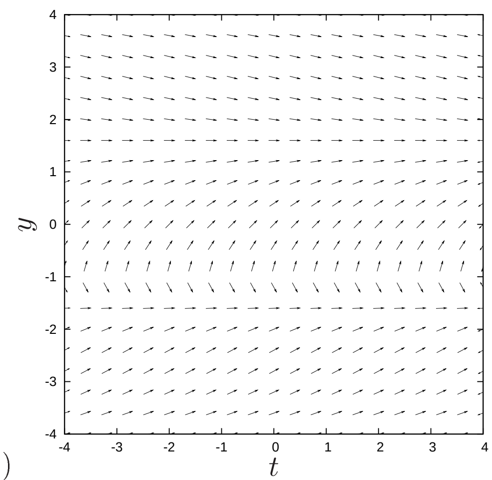
   (1) $\dfrac{dy}{dt} = t\sin(y)$, (2) $\dfrac{dy}{dt} = y\exp\left(-\dfrac{t^2}{2}\right)$, (3) $\dfrac{dy}{dt} = \cos(y)/(y+1)$, (4) $\dfrac{dy}{dt} = \dfrac{1}{2}ty + t$



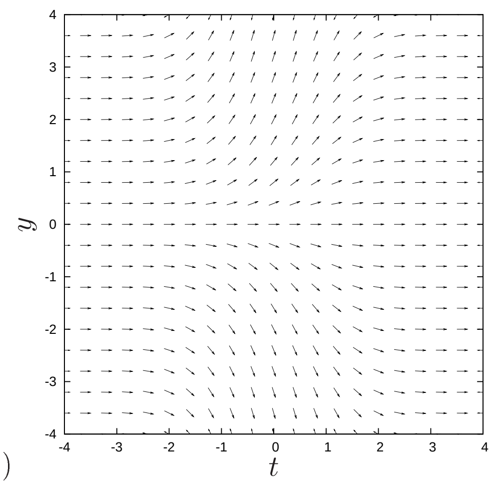
(A)



(B)



(C)



(D)

5. **(10 points)** Consider the direction field plotted on the supplemental sheet given with this exam, which is for a first-order differential equation (*i.e.*, $\dfrac{dy}{dt} = f(t,y)$). For the initial condition $y(0) = 1$, sketch the approximate solution for $0 \le t \le 8$ that would be produced with Euler's method with $\Delta t = 1$. Be sure to use a straight edge.

Below is the supplemental figure, which should be submitted with your bluebook. Be sure to write your name on this page.