UC Merced: MATH 24 — Final Exam — 26 October 2006

On the front of your bluebook print (1) your name, (2) your student ID number, (3) your instructor's name (Sprague) and (4) a grading table. Show all work in your bluebook and BOX IN YOUR FINAL ANSWERS where appropriate. A correct answer with no supporting work may receive no credit while an incorrect answer with some correct work may receive partial credit. Textbooks, class notes, and calculators are not permitted. You are permitted two handwritten 8.5×11 inch crib sheets. There are a total of 12 problems and a total of 150 points. Please start each of the 12 problems on a new page.

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1. **(25 points)** Answer the following Always True or False. Only your final **boxed** answer will be graded on these problems, and you must write out the words TRUE or FALSE completely.

   (a) Consider the differential equation $\frac{dy}{dt} = f(t)$, where

   $$f(t) = \begin{cases} \cos(t) & t \geq 0 \\ 1 & t < 0 \end{cases}$$

   Picard's theorem guarantees that there exists a unique solution for the initial condition $y(-1) = 2$ in some region around $t = -1$.

   (b) If $L(y)$ is a linear operator, and if y_1 and y_2 are solutions to $L(y) = f(t)$ for some function $f(t)$, then $y_3 = y_1 + y_2$ is also a solution, i.e., $L(y_3) = f(t)$.

   (c) The vectors $\{(1,4)^T, (2,6)^T, (-3,-9)^T\}$ span $\mathbb{R}^2$.

   (d) If B is as given below, then $\det B = 20$.

   $$B = \begin{bmatrix} 0 & -5 & 6 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

   (e) Consider the system of differential equations for $x_1(t)$ and $x_2(t)$:

   $$\begin{aligned} \frac{dx_1}{dt} &= x_1 - x_2 \\ \frac{dx_2}{dt} &= x_1^2 - x_2 \end{aligned}$$

   The point $x_1 = x_2 = 1$ is an unstable equilibrium point.

2. **(10 Points)** Use *separation of variables* to solve for $y(t)$ given

   $$\frac{dy}{dt} = y\frac{\cos(\ln(t))}{t}$$

3. **(10 points)** Consider the direction field plotted on the supplemental sheet given with this exam, which is for a first-order differential equation (*i.e.*, $\frac{dy}{dt} = f(t, y)$).

   (a) For the initial condition $y(0) = 4$, sketch the approximate solution for $0 \leq t \leq 8$ that would be produced with Euler's method with $\Delta t = 1$. Be sure to use a straight edge.

   (b) Sketch, to the best of your abilities, the ODE solution trajectory for the initial condition $y(0) = 4$.

4. **(10 points)** Using the classifications discussed in this course, classify the following differential equations for $y(t)$ as thoroughly as you can.

   (a) $\frac{d^2y}{dt^2} + y = \cos(y)$

   (b) $\frac{d^3y}{dt^3} + t(1 - y) = 0$

5. **(10 points)** Do the following vectors constitute a basis for $\mathbb{R}^3$? Be sure to justify your answer.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

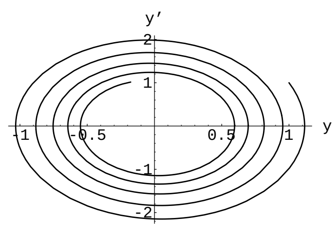
6. **(10 points)** Consider the set $\mathbb{W} = \{f(t) | \frac{d^2f}{dt^2} + p(t)\frac{df}{dt} + q(t)f = 0, \forall p, q \in \mathcal{C}^0\}$. Given that $\mathbb{W}$ is a subset of $\mathcal{C}^2$ (the set of all continuous functions with 2 continuous derivatives), determine if $\mathbb{W}$ constitutes a **subspace** of $\mathcal{C}^2$.

7. **(10 points)** Determine the general solution $y(t)$ given

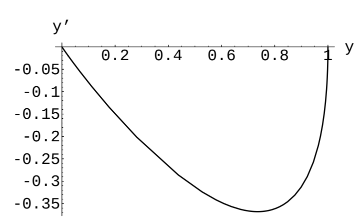
$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = \cos(t)$$

8. **(10 points)** Match each of the two differential equations with the appropriate phase-plane trajectory for the initial conditions $y(0) = 1$, $y'(0) = 0$. Only your final answer will be graded on this problem (A,B,C, or D for each equation).

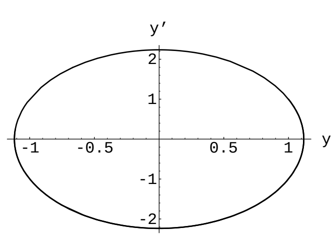
$$(1) \quad \frac{d^2y}{dt^2} + 4y = 0, \qquad (2) \quad \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0,$$



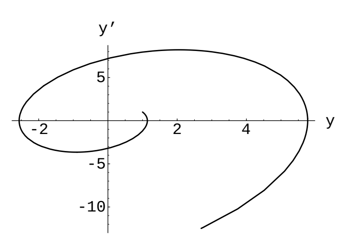
(A)



(B)



(C)



(D)

9. **15 Points** When we discussed *Variation of Parameters* in class, we assumed that the particular solution to

$$y'' + p(t)y' + q(t)y = f(t),$$

was $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$ where $y_1(t)$ and $y_2(t)$ are linearly independent solutions to the homogeneous equation, and $v_1(t)$ and $v_2(t)$ need to be determined. We also placed the restriction that $v_1(t)$ and $v_2(t)$ are such that

$$v_1'(t)y_1(t) + v_2'(t)y_2(t) = 0, \tag{1}$$

which resulted in two equations in the two unknowns $v_1'(t)$, $v_2'(t)$:

$$\begin{aligned} v_1'(t)y_1(t) + v_2'(t)y_2(t) &= 0, \\ v_1'(t)y_1'(t) + v_2'(t)y_2'(t) &= f(t). \end{aligned}$$

What are the resulting two equations in the two unknowns $v_1'(t)$, $v_2'(t)$ that result if we replace the restriction in Eq. (1) (for some odd reason) with

$$v_1'(t)y_1(t) + v_2'(t)y_2(t) = 2,$$

assuming that $y_1(t)$ and $y_2(t)$ are known?

10. **(10 points)** Determine the eigenvalues and eigenvectors of the matrix A where

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

11. **(10 points)** Consider the system of differential equations $\mathbf{x}' = A\mathbf{x}$ for $\mathbf{x}^T(t) = [x_1(t), x_2(t)]$ where,

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

Given that the eigenvalues are $\lambda_{1,2} = 1 \pm 2i$ and the eigenvector associated with λ_1 is $\mathbf{v}_1^T = [-i, 1]$, write the general solution.

12. **(20 points total)** Consider the system of differential equations for $x_1(t)$ and $x_2(t)$:

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 - x_2 \\ \frac{dx_2}{dt} &= x_1^2 - x_2 \end{aligned}$$

   (a) (10 points) Determine the equilibrium points, and determine the equations governing the nullclines on the $x_2(t)$ vs. $x_1(t)$ plane

   (b) (10 points) On the supplemental figure (page 5), sketch the equilibrium points (as circles) and nullclines. Draw appropriate arrows on the nullclines to indicate the direction of solution curves.

Supplemental Sheet to be Submitted with Blue Book

Student Name:

Figure for Problem 3

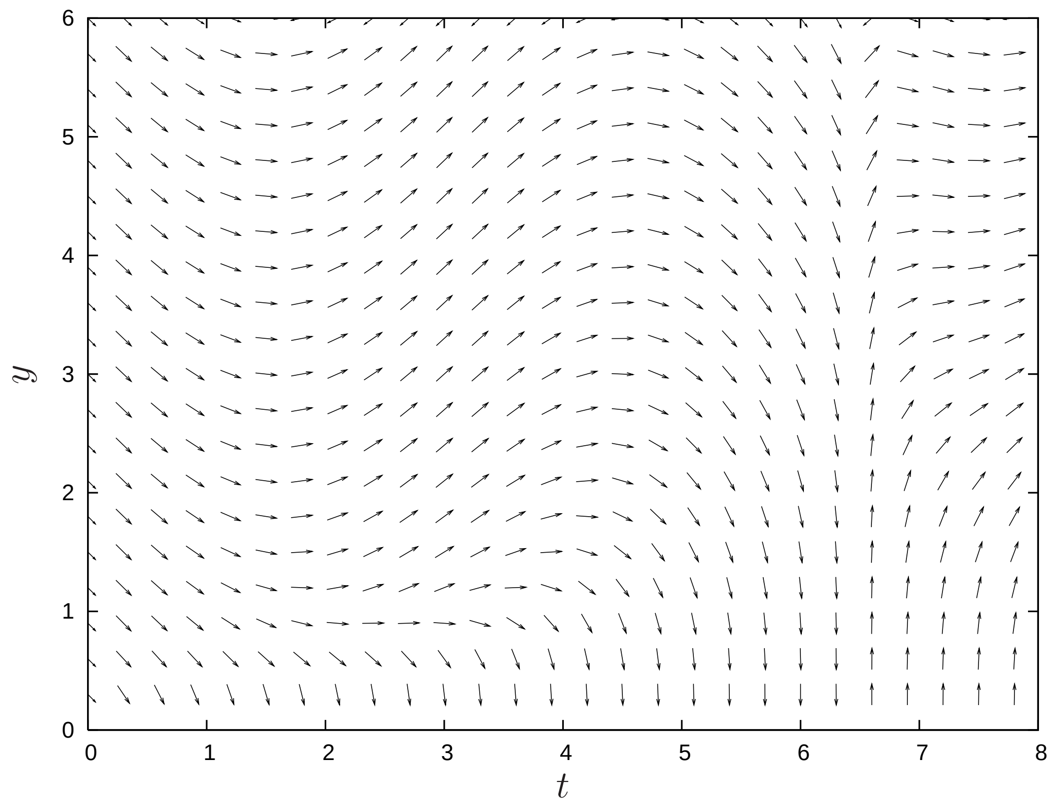


Figure for Problem 12