Math 24

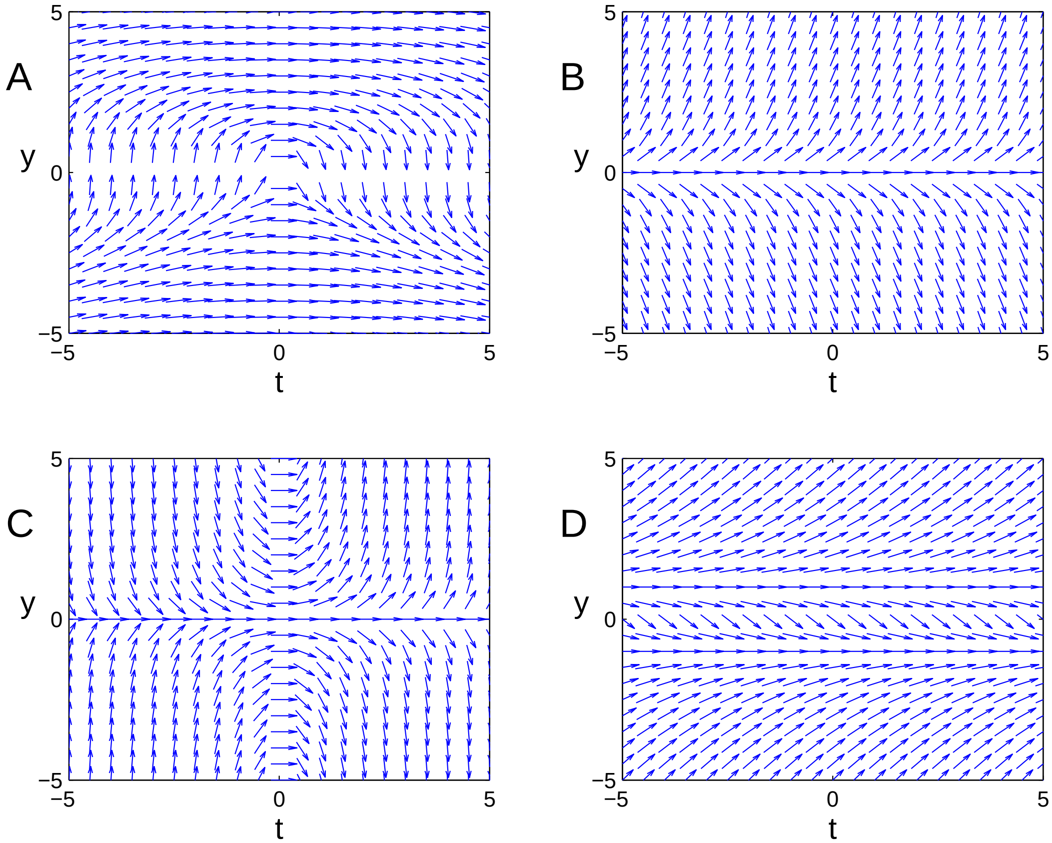
Exam 1: February 15, 2006

ON THE FRONT OF YOUR BLUEBOOK WRITE (1) YOUR NAME, (2) A FIVE-PROBLEM GRADING GRID. **Show ALL of your work** in your bluebook, and **box in your final answers.** A correct answer, but without the relevant work, will receive no credit. This exam is closed-book and no calculators are allowed. You are allowed a one-page crib sheet. Start each problem on **the top of a new page.** Each problem is worth 20 points, for a total of 100 points. You can solve the problems in any order you like.

1. (a) Match the following differential equations (1)–(4) with their corresponding direction fields A–D.

   (1) $\frac{dy}{dt} = y + \sin(y)$, (2) $\frac{dy}{dt} = +\sqrt{|y|} - 1$, (3) $\frac{dy}{dt} = -\frac{t^2}{y}$, (4) $\frac{dy}{dt} = ty$.

   (b) Do equations (1)–(4) have any equilibrium solutions? If so, find the equilibrium solutions for each equation.



SOLUTION:

(a) (1,B), (2,D), (3,A), (4,C)

(b) Equation 1: $y = 0$; Equation 2: $y = \pm 1$; Equation 3: none; Equation 4: $y = 0$;

2. Classify the following equations as (1) linear or nonlinear, and (2) separable or non-separable. Then solve these equations.

   (a) $\sin(t)y' + \cos(t)y = 1$.

   (b) $y' = \dfrac{t(t+1)}{y^2}$.

SOLUTION:

(a) This is a non-separable linear equation with the standard linear form $y' + \dfrac{\cos(t)}{\sin(t)}y = \dfrac{1}{\sin(t)}$. This yields the integrating factor $\mu(t) = e^{\int \frac{\cos(t)dt}{\sin(t)}} = e^{\ln|\sin(t)|} = \sin(t)$. After multiplication by $\mu(t)$ we get back the original equation – which can be written in the form $\dfrac{d[\sin(t)y]}{dt} = 1$. Integrating, we have $\sin(t)y = t + c$ or $y(t) = \dfrac{t+c}{\sin(t)}$.

(b) This is a nonlinear separable equation. Separating variables and integrating $\int y^2\,dy = \int (t^2+t)\,dt$ yields $\dfrac{y^3}{3} = \dfrac{t^3}{3} + \dfrac{t^2}{2} + c$, or $y(t) = t^3 + 1.5t^2 + c$.

3. Consider the equation

$$y' = 2y^2(y^2 - a^2) \ ,$$

where a is a positive constant ($a > 0$).

(a) Find the equilibrium solutions.

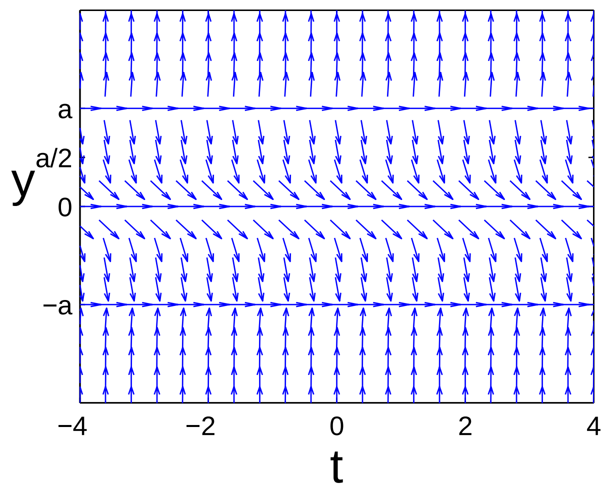
(b) Sketch the phase lines and direction fields.

(c) Determine the stability of the equilibrium solutions.

(d) Find the limiting value of $y(t)$ as $t \to \infty$ when $y(0) = \frac{a}{2}$. Justify your answer.

SOLUTION:

(a) First, it helps to rewrite the equation as $y' = 2y^2(y + a)(y - a)$. It follows that the equilibrium solutions are $y = 0, \pm a$.



(b)

(c) $y = -a$ is stable, $y = +a$ is unstable, and $y = 0$ is semi-stable (stable from above and unstable from below).

(d) Since the equilibrium solution $y = a$ is unstable and $y = 0$ is stable from above, the solution with $y(0) = \frac{a}{2}$ will approach $y = 0$ for long time. Note that the solution cannot "pass through" $y = 0$, since that would violate the uniqueness property of this IVP.

4. Determine whether the following statements are TRUE or FALSE. Note: you must write the entire word TRUE or FALSE. You do not need to show your work for this problem.

(a) $y(x) = x\cos(x)$ is a solution of the IVP

$$y' - \frac{y}{x} = -x\sin(x)\ , \quad y(0) = 0\ .$$

(b) Picard's theorem guarantees the local existence and uniqueness of a solution to the IVP

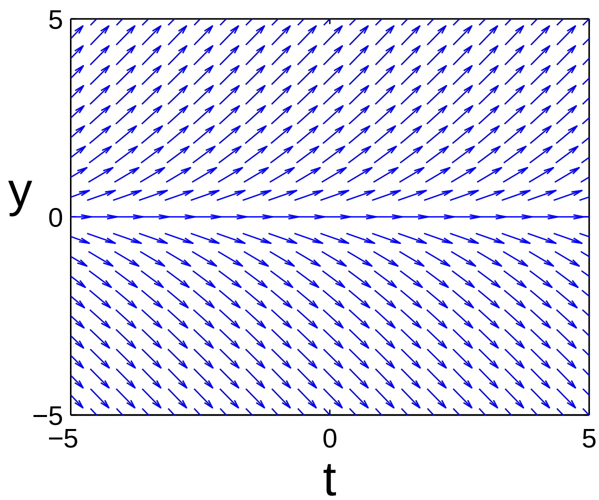
$$\frac{dy}{dx} = \tan^{-1}(y)\ , \quad y(0) = c\ ,$$

for all values of c.

SOLUTION:

(a) TRUE

(b) TRUE. Here $f(t, y) = \tan^{-1}(y)$ is continuous for all values of (t, y), hence the solution exists for all values of c. In addition, $\dfrac{df}{dy} = \dfrac{1}{1+y^2}$ is also continuous for all values of (t, y), hence the solution is unique for all values of c.
Note: The fact that $\tan^{-1}(y)$ is bounded between $-\pi/2$ and $\pi/2$ should not be confused with the initial condition being any value of $y(0)$!! If you are still confused, look at the direction field below and convince yourself that (1) there are no singularity points and (2) the slopes reach $\pm\pi/2 \approx \pm 1.6$ on the top and bottom.



5. Solve the equation

$$\frac{dy}{dt} - \frac{y}{t} = \left(\frac{y}{t}\right)^2 .$$

Hint: since this is an *Euler-homogeneous equation*, you can use $v = \dfrac{y}{t}$ to transform it into a separable equation. Alternatively, since this is also a *Bernoulli equation*, you can use $v = y^{-1}$ to transform it into a linear equation.

SOLUTION:

(a) As an Euler-homogeneous equation define $v = \dfrac{y}{t}$. Then $y = vt$, so $y' = v't + v$. Substituting this into the equation yields $(v't + v) - v = v^2$ or $v' = \dfrac{v^2}{t}$. Separating variables we get

$$\frac{dv}{v^2} = \frac{dt}{t} .$$

Integrating, we have

$$-\frac{1}{v} = \ln(t) + c .$$

Therefore, $v = \dfrac{1}{c - \ln(t)}$ and $y = vt = \dfrac{t}{c - \ln(t)}$.

(b) As a Bernoulli equation you can divide by y^2 to get

$$y^{-2}\frac{dy}{dt} - \frac{1}{yt} = \frac{1}{t^2} .$$

Defining $v = y^{-1}$ and using the chain rule we have $\dfrac{dv}{dt} = -\dfrac{y'}{y^2}$. Transforming the equation from y to v yields

$$-\frac{dv}{dt} - \frac{v}{t} = \frac{1}{t^2} ,$$

which is linear equation in v, whose standard form is

$$\frac{dv}{dt} + \frac{v}{t} = -\frac{1}{t^2} .$$

Multiplying by the integrating factor $\mu(t) = t$ gives

$$\frac{d(tv)}{dt} = -\frac{1}{t} .$$

After integrating we get $v(t) = \dfrac{c - \ln(t)}{t}$ and, therefore, $y(t) = \dfrac{t}{c - \ln(t)}$.

THE END