Duration: 3 hours
Instructions: Answer all questions, without the use of notes, books or calculators. Partial credit will be awarded for correct work, unless otherwise specified. The total number of points is 100.

1. (10 pts) Consider the differential equation $y'(t) = (t+2)(y^2-9)$ for $t > 0$.
   (a) Find all the equilibrium points of this equation and classify them as stable or unstable.
   (b) For which initial conditions, if any, may the solution not be unique?
   (c) Find the general solution to this equation using the identity below. You do not need to solve for $y(t)$ explicitly so your answer may be of the form $f(y) = g(t)$.
$$\frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$$
2. (9 pts) Find the solution to the initial value problem $y'(t) - y/t = 2t$ with $y(1) = 5$.
3. (8 pts) In the absence of predators, the population of koalas, K, doubles every 5 years. However, each Tasmanian devil eats 24 koalas per year. The population of Tasmanian devils, T, increases at a yearly rate given by the population of Tasmanian devils multiplied by the difference between the number of koalas and the number of Tasmanian devils. Recently, a boat with 200 koalas and 30 Tasmanian devils on board was stranded on Catalina island, thus introducing the first koalas and Tasmanian devils to the island (no animals were hurt in the shipwreck!)

   Using t to represent the number of years elapsed since the shipwreck, find differential equations with appropriate initial conditions describing the populations of koalas, K, and of Tasmanian devils, T.
4. (10 pts) Consider the following matrix and vector
$$A = \begin{bmatrix} 1/2 & 0 & 2 \\ -3 & 2 & -4 \\ 1 & 1 & 8 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} -1 \\ 2 \\ -4 \end{bmatrix}$$
   a) Find all the solutions to $A\vec{x} = \vec{b}$. If no solutions exists, explain why.
   b) What is the Kernel of the transformation $T(\vec{x}) = A\vec{x}$?
5. (10 pts) Find the general solution of $x'' - 8x' + 16x = t + 2$.
6. (8 pts) Consider systems of equations of the form $x_1' = ax_1 + bx_2$, $x_2' = cx_1 + dx_2$, with a, b, c, d some real numbers. Give possible eigenvalues and eigenvectors **AND** sketch a corresponding phase portrait such that:
   (a) The origin is an unstable equilibrium point and has at least one complex eigenvalue.
   (b) The origin is a stable equilibrium point.
7. (8 pts) Find all the eigenvalues and eigenvectors of the matrix A given below.
$$A = \begin{bmatrix} 4 & -3/2 \\ 7/2 & -1 \end{bmatrix}$$

8. (9 pts) Find the general solution to $\vec{x}' = A\vec{x}$, with A given below, using the fact that the eigenvectors of A are $\vec{v_1} = (1, 2i, 0)^T$, $\vec{v_2} = (1, -2i, 0)^T$ and $\vec{v_3} = (1, -1, 2)^T$. You may express your answer as the product of complex vectors and complex functions. For extra credit (2 pts), write your answer in terms of real functions only.

$$A = \begin{bmatrix} 2 & 1 & 3/2 \\ -4 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

9. (8 pts) Consider the nonlinear system of differential equations given below.
   (a) Find all its equilibrium points.
   (b) Find a linear system of equation approximating the solutions in a neighborhood of the equilibrium point closest to the origin.

$$\frac{dx_1}{dt} = x_1^2 - x_2 x_1 - 2x_2 - x_1 - 6$$
$$\frac{dx_2}{dt} = \frac{1}{2x_2} - \frac{1}{x_1}$$

10. (20 pts) Answer the following questions in no more than two lines of text (much less is actually needed if you are right on point). Minimal (if any) computation is required.
    (a) A given equation $y'(t) = f(y)$, with $f(y)$ a continuous function, has only a stable equilibrium at $y = 4$ and an unstable equilibrium at $y = 0$. If $y_1(t)$ is a solution of this equation and satisfies $y_1(0) = 1$, what can you say about $\lim_{t\to\infty} y_1(t)$?
    (b) For what differential equation is the $n+1$ iteration of Euler's method given by $\tilde{y}_{n+1} = \tilde{y}_n + t_n^2(\tilde{y}_n + 1)\Delta t$?
    (c) Give an example of a third order, linear, homogenous differential equation with non-constant coefficients.
    (d) If you know that span$\{\vec{v_1}, \vec{v_2}, \vec{v_3}\} = \mathbb{P}$, what conditions must the vectors $\vec{v_1}$, $\vec{v_2}$ and $\vec{v_3}$ satisfy to form a basis of $\mathbb{P}$?
    (e) How would you verify that a given matrix B is the inverse of another given matrix A?
    (f) Give an example of a *linear* transformation of functions of one variable $L(f(t))$.
    (g) Sketch the time evolution of a harmonic oscillator forced at a frequency that is close but not equal to its natural frequency and which therefore exhibits beats.
    (h) What do you assume is the form of a particular solution when using the method of variation of parameters to solve an equation of the form $y' + p(t)y = q(t)$? Explain your notation.
    (i) Give an example of three vectors $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$ such that no vector is a multiple of another but for which $\vec{v_3}$ is linearly dependent on $\vec{v_1}$ and $\vec{v_2}$.
    (j) Give a physical example of a stable equilibrium point and a physical example of an unstable equilibrium point.

Enjoy your remaining exams... and your break too!