Math 24 Spring 2009 - Exam 2 Solutions

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1. (35 points) Answer the following TRUE/FALSE questions. You must write the entire word TRUE or FALSE.
   (a) The vectors $\{[1,0,1],[2,0,2],[0,1,0]\}$ are linearly independent
   (b) The dimension of the span of $\{[1,0,1],[2,0,2],[0,1,0]\}$ is 3.
   (c) The functions $\{1, 1+t, 2+t\}$ are linearly independent.
   (d) If $A\mathbf{x} = 0$ has a solution $\mathbf{x} \neq 0$ then $A\mathbf{x} = \mathbf{b}$ must have infinitely many solutions for any $\mathbf{b}$.
   (e) Four vectors in $\Re^3$ are necessarily linearly dependent.
   (f) Four vectors in $\Re^3$ necessarily span $\Re^3$
   (g) The set of solutions of the differential equation $y' + ty = t$ forms a vector space.

SOLUTION:

(a) For linear independence, no vector should be a linear combination of the others. However, the vector $[2,0,2]$ is $2\times[1,0,1]$. So the statement is FALSE.

(b) These are the same vectors as in part (a). Since two of them are linearly dependent the dimension of the span of all three of them is not 3, but 2. Hence, the statement is FALSE.

(c) If you notice that the third vector is the sum of the first two you may conclude that they are linearly dependent and the statement is FALSE. Alternatively, we could perform the Wronskian test. Here, we find that the determinant of the Wronskian matrix is 0, so they are linearly dependent.

(d) $A\mathbf{x} = 0$ always admits the (trivial) solution $\mathbf{x} = 0$. When $A\mathbf{x} = 0$ has a non-trivial solution the matrix A must be singular. In that case**,** $A\mathbf{x} = \mathbf{b}$ may **either** have no solution or infinitely many, depending on the vector **b.** Since this vector is not specified in the question the answer is FALSE.

(e) The number of vectors is greater than the dimension. Thus, they are linearly dependent and the statement is TRUE.

(f) Four vectors need not necessarily span this space. For example, if all four vectors are the same. So the statement is FALSE.

(g) The differential equation is linear, but **NOT** homogeneous. For the set of solutions of a differential equation to form a vector space, the differential equation must be linear **AND** homogeneous, so the statement is FALSE.

2. (32 points) Consider the linear system of equations

$$-x + y - z = 42$$
$$x + y + z = 0$$

(a) Find the solutions of this system using Gaussian Elimination.
(b) What do the solutions of the above system span geometrically (a point, line, plane, ...)?
(c) What is the definition <u>in general</u> of a *basis* of a vector space?
(d) Determine whether the solutions of the system above form a basis for $\Re^3$.

SOLUTION:

(a) Setting our system in the augmented form, we perform Gaussian Elimination operations:

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 42 \\ 1 & 1 & 1 & 0 \end{array}\right] \overset{\substack{R_1=-R_1 \\ R_2=-R_2}}{\rightarrow} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -42 \\ -1 & -1 & -1 & 0 \end{array}\right] \overset{R_2=R_2+R_1}{\rightarrow}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -42 \\ 0 & -2 & 0 & -42 \end{array}\right] \overset{R_2=-\frac{R_2}{2}}{\rightarrow} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -42 \\ 0 & 1 & 0 & 21 \end{array}\right] \overset{R_1=R_1+R_2}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -21 \\ 0 & 1 & 0 & 21 \end{array}\right].$$

So we have that $\begin{cases} x + z = -21 \\ y = 21 \end{cases}$

If we let $t = z$ we get the solutions $(-21-t, 21, t)$. There is one degree of freedom so the solutions span a line. This should not be too surprising, because each of the two equations corresponds to a plane and two planes typically (but not always) intersect at a line.

(b) The basis of a vector space is a set of vectors such that they span a vector space and they are linearly independent.

(c) In part (b) we obtained that the solutions span a line, which is a (proper) subspace of $\Re^3$. Hence, they cannot span $\Re^3$ and therefore cannot be a basis for it.

3. (30 points) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & -1 \\ 0 & 1 & k \end{bmatrix}$ where k is a real constant, and let $\mathbf{b} = \begin{bmatrix} 42 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

(a) Calculate the determinant of A. For which values of k does A^{-1} exist?
(b) Find A^{-1} when $k = 0$.
(c) Find the solutions of $A\mathbf{x} = 0$ when $k = 0$.
(d) Which of the following operations are defined (short answers): (i) $A\text{-}9I$, (ii) $|Ab|$, (iii) $\mathbf{b}^T + \mathbf{b}$, (iv) $(\mathbf{b}^T A)^T$, (v) $\mathbf{b}^T \mathbf{b}$, (vi) $\mathbf{b}\mathbf{b}^T$?
(e) Find an example of two 2x2 matrices, A and B such that $AB \neq BA$

SOLUTION:

(a) The determinant is $k^2 + 1$. In general, the inverse of A exists if and only if its determinant is nonzero. However, $k^2 + 1$cannot be negative or zero for any real k. Therefore, the inverse exists for all real k.

(b) To find the inverse, we can use the method of Gaussian Elimination (RREF) as follows.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{R_2=R_3 \\ R_3=R_2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_3=-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{array}\right]$$

So the inverse is $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$.

(c) We could use the method of Gaussian Elimination again. However, as we have already obtained in part (a) that A is invertible (in particular when $k = 0$) we conclude that $A\mathbf{x} = 0$ admits only the trivial solution $\mathbf{x} = 0$.

(d) (i) $A\text{-}9I$; <u>DEFINED</u>. The dimensions are the same for both matrices for subtraction.

(ii) $|Ab|$; <u>NOT DEFINED</u>. The product of Ab is not defined since the number of row elements of A and the column elements of $\mathbf{b}$ are not the same. In addition, you can only take the determinant of a square matrix.

(iii) $\mathbf{b}^T + \mathbf{b}$; <u>NOT DEFINED</u>. The transpose and the original $\mathbf{b}$ vectors have different dimensions. Hence, their addition is not defined.

(iv) $(\mathbf{b}^T A)^T$; $(\mathbf{b}^T A)^T = A^T \mathbf{b}$. <u>UNDEFINED</u>. The product of $A^T \mathbf{b}$ is not defined since the number of row elements of A^T and the column elements of $\mathbf{b}$ are not the same. So the product is undefined.

(v) $\mathbf{b}^T\mathbf{b}$; <u>DEFINED</u>. The number of row elements of $\mathbf{b}^T$ and column elements of $\mathbf{b}$ are one and the same. The resulting product is a scalar (number).

(vi) $\mathbf{b}\mathbf{b}^T$; <u>DEFINED</u>. The number of row elements of $\mathbf{b}$ and column elements of $\mathbf{b}^T$ are one and the same. The resulting product is a 4 x 4 matrix.

(e) In general, the product of any two matrices is not commutative (i.e. *AB* is not the same as *BA*). In fact, if you choose "random" looking matrix elements most likely the resulting matrices will not commute. For example,

$$A = \begin{bmatrix} 24 & 3 \\ 23 & 34 \end{bmatrix} \ \& \ B = \begin{bmatrix} 2 & 34 \\ 48 & 24 \end{bmatrix}$$