SOLUTION

Midterm 3: Math 30, 11/19/07

4 pts 1) Find the length of the curve.

$y = 4x^{3/2}, 0 \le x \le 1$

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$y' = 6x^{1/2} \qquad (y')^2 = 36x$$

$$L = \int_0^1 \sqrt{1+36x}\, dx \qquad u = 1+36x \qquad du = 36\, dx$$

$$L = \frac{1}{36}\int_1^{37} u^{1/2}\, du = \frac{1}{36}\,\frac{2}{3}\, u^{3/2}\Big|_1^{37} = \frac{1}{54}\left(37\sqrt{37} - 1\right)$$

4 pts 2) Find the solution of the differential equation $(x^2+1)\frac{dy}{dx} = xy$ that satisfies the initial condition $y(1) = 1$.

$u = x^2+1, \; du = 2x\,dx$

$$\frac{dy}{dx} = \frac{xy}{(x^2+1)} \qquad \int \frac{dy}{y} = \int \frac{x}{x^2+1}\, dx =$$

$$\ln y = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln u = \frac{1}{2}\ln|x^2+1| + C$$

Take exponent of both sides. $e^{a+b} = e^a e^b$

$$y(x) = e^C (x^2+1)^{1/2} = A(x^2+1)^{1/2} \qquad A = e^C$$

$$y(1) = 1$$

$$1 = A(2)^{1/2} \qquad A = \frac{1}{\sqrt{2}}$$

$$\boxed{y(x) = \frac{1}{\sqrt{2}}(x^2+1)^{1/2}}$$

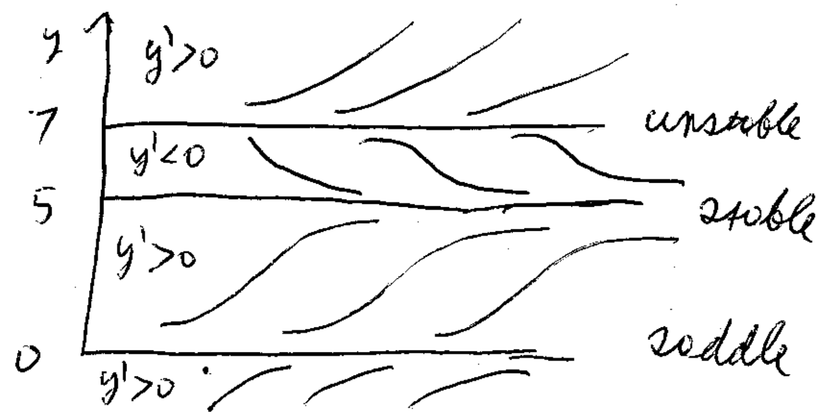
4 pts 3) A function $y(t)$ satisfies the differential equation. $\frac{dy}{dt} = y^4 - 12y^3 + 35y^2$

a) Find and plot equilibrium points

b) Determine whether equilibrium points are stable, unstable or saddle.

$$\frac{dy}{dt} = y^2(y^2 - 12y + 35) = y^2(y-7)(y-5)$$

Equilibrium points $\frac{dy}{dt} = 0$ occur when $y = 0, 5, 7$



4 pts 4) Find the area of the surface obtained by rotating the curve about the x-axis.

$y = (4 - x^2)^{1/2}, 0 \le x \le 1$

$$S = \int 2\pi y \, ds = \int 2\pi y \sqrt{1 + (y')^2} \, dx$$

$$y' = \tfrac{1}{2}(4-x^2)^{-1/2}(-2x) = -x(4-x^2)^{-1/2}$$

$$(y')^2 = \frac{x^2}{4-x^2}$$

$$S = 2\pi \int_0^1 (4-x^2)^{1/2} \left(1 + \frac{x^2}{4-x^2}\right)^{1/2} dx$$

$$S = 2\pi \int_0^1 (4 \cancel{-x^2} \cancel{+x^2})^{1/2} dx = 2\pi \int_0^1 \sqrt{4}\, dx$$

$$= \boxed{2\pi \cdot 2x \Big|_0^1 = 4\pi}$$

4 pts 5) A bacteria culture grows at the rate of $\frac{dP}{dt} = \lambda P$.

a) Find the solution for P(t) given the initial condition P(0) = P_o (You can guess the solution but show that it satisfies the differential equation)

b) How long does it take for the initial population to double?

a) This is exponential growth equation

$P(t) = P_0 e^{\lambda t}$ $\quad \frac{dP}{dt} = \lambda P_0 e^{\lambda t} = \lambda P$

b) When population doubles $P(t_2) = 2P_0$

$2\cancel{P_0} = \cancel{P_0} e^{\lambda t_2}$ $\quad \ln 2 = \lambda t_2$

$$\boxed{t_2 = \frac{\ln 2}{\lambda}}$$

Extra Credit (4 pts)

Solve the linear differential equation given the initial condition y(0)=2

$y' = xe^{-\sin x} - y\cos x$

Rewrite as $\frac{dy}{dx} + \cos x \, y = x e^{-\sin x}$

$P(x) = \cos x$

Integration factor $\rho(x) = \exp \int \cos x \, dx = \exp(\sin x)$

$\rho\left(\frac{dy}{dx} + \cos x \, y\right) = x$ Multiply by the integration factor

$\frac{d}{dx}(\rho y) = x$ $\quad \rho y = \int x \, dx$

$\rho y = \frac{x^2}{2} + C$

$y(x) = \frac{x^2}{2} e^{-\sin x} + C e^{-\sin x}$ $\quad y(0) = 2$

$2 = C$

$$\boxed{y(x) = \frac{x^2}{2} e^{-\sin x} + 2e^{-\sin x}}$$