Final Exam, Math 30, December 15, 2007

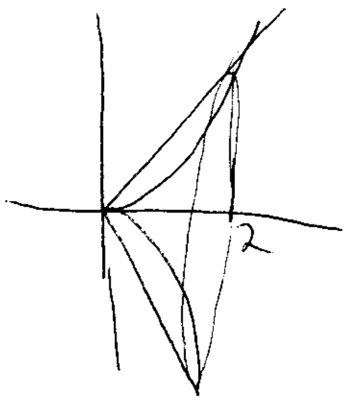
Solve for y explicitly. Indefinite integrals must be in terms of x. Check out useful information in the Appendix.

1) Find the volume of the solid obtained by rotating the given region about the x-axis. **(10 pts)**

Region bounded by: $y = 2x, y = x^2$

Curves intersect at $2x = x^2$ or $x = 2$

Slice the region into washers



$dV = \pi r_1^2 - \pi r_2^2 = \pi(4x^2 - x^4)$

$V = \int dV = \int_0^2 \pi(4x^2 - x^4)\,dx$

$= \frac{4\pi x^3}{3} - \frac{\pi x^5}{5}\Big|_0^2 = \pi\left(\frac{32}{3} - \frac{32}{5}\right)$

$\boxed{= \frac{64\pi}{15}}$

2) Find the average value of the function $u(x) = 10x\sin(x^2)$ on the interval $\left[0, \sqrt{\pi}\right]$ **(10 pts)**

$u_{AV} = \frac{1}{\sqrt{\pi} - 0}\int_0^{\sqrt{\pi}} 10x \sin x^2\,dx$ let $x^2 = u$, $2x\,dx = du$

$u_{AV} = \frac{1}{\sqrt{\pi}}\int_0^{\pi} 5 \sin u\,du = -\frac{1}{\sqrt{\pi}} 5\cos u\Big|_0^{\pi} = \frac{5}{\sqrt{\pi}} + \frac{5}{\sqrt{\pi}}$ $\cos\pi = -1$

$\boxed{= \frac{10}{\sqrt{\pi}}}$

Evaluate the following indefinite integrals **(10 pts each)** solutions must be in terms of x.

3) $\int \sin^2(x)\cos^3(x)dx$

$$\int \sin^2(x)\cos^2(x)\cos x\,dx \qquad u = \sin x,\quad du = \cos x\,dx,\quad \cos^2 x = 1-\sin^2 x$$

$$\int u^2(1-u^2)\,du = \int u^2 - u^4\,du.$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C = \boxed{\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C}$$

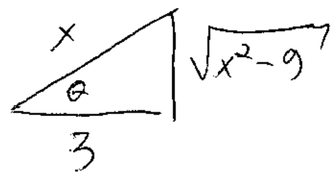
4) $\int \frac{1}{x^2\sqrt{x^2-9}}dx$

Let $x = 3\sec\theta$ $\qquad \sec^2\theta - 1 = \tan^2\theta$

$dx = 3\sec\theta\tan\theta$

$$\int \frac{3\sec\theta\tan\theta\,d\theta}{9\sec^2\theta\sqrt{9(\sec^2\theta-1)}} = \int \frac{d\theta}{9\sec\theta} = \frac{1}{9}\int \cos\theta\,d\theta$$

$$= \frac{1}{9}\sin\theta$$



$$\boxed{= \frac{1}{9}\,\frac{\sqrt{x^2-9}}{x} + C}$$

5) $\int \frac{x+1}{x^2+5x+6}dx$ partial fractions method

$$\frac{x+1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$x+1 = A(x+3) + B(x+2)$$

let $x=-3 \Rightarrow -2 = -B(1)$ $\boxed{B=2}$

$x=-2$ $\boxed{-1 = A}$

$$\int \frac{x+1}{x^2+5x+6}dx = -\int \frac{dx}{x+2} + 2\int \frac{dx}{x+3}$$

$$\boxed{= -\ln|x+2| + 2\ln|x+3| + C}$$

6) Solve the differential equation that satisfies the given initial condition. (10 pts)

$\frac{dy}{dx} = y^2 \sin x$, $y(\frac{\pi}{2}) = 2$ use the separation of variables method

$$\int \frac{dy}{y^2} = \int \sin x \, dx$$

$$-y^{-1} = -\cos x + C$$

$$y = \frac{1}{\cos x - C}$$

$y(\frac{\pi}{2}) = 2$ $2 = \frac{1}{-C}$

$C = -\frac{1}{2}$

$$\boxed{y(x) = \frac{1}{\cos x + \frac{1}{2}}}$$

7) Solve the differential equation: $\frac{dy}{dx} = x + y$ where y(0)=1.0 **(10 pts)**

This is a linear first order equation. Use the integration factor technique to solve it.

$$\frac{dy}{dx} - y = x$$

$$p(x) = -1$$

$$g(x) = x$$

$$I(x) = \exp(\int(-1)dx) = \exp(-x)$$

$$y = \frac{\int I(x)g(x)}{I(x)} = \frac{\int x\exp(-x)dx}{\exp(-x)}$$

$$\int x\exp(-x) =$$

$$u = x \qquad dv = \exp(-x)$$

$$du = dx \quad v = -\exp(-x)$$

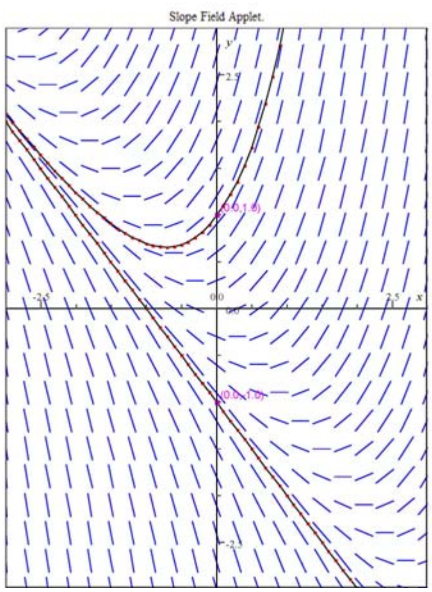
$$= -x\exp(-x) + \int \exp(-x) = -x\exp(-x) - \exp(-x) + c$$

$$y(x) = -x - 1 + c\exp(x)$$

$$y(0) = 1 \Rightarrow c = 2$$

$$y(0) = -1 \Rightarrow c = 0$$

Sketch a graph of the solutions that satisfy the given initial conditions on the slope field map. **(5 pts)**



8) A function y(t) satisfies the differential equation:

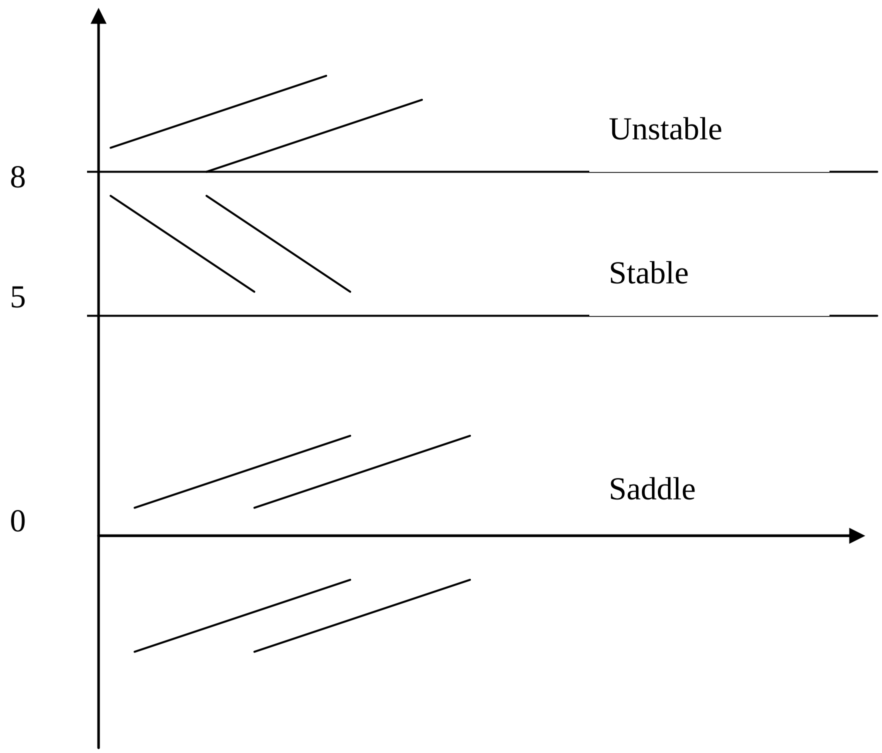
$$\frac{dy}{dt} = y^4 - 13y^3 + 40y^2$$

a) Find and plot equilibrium points **(10 pts)**

2.0

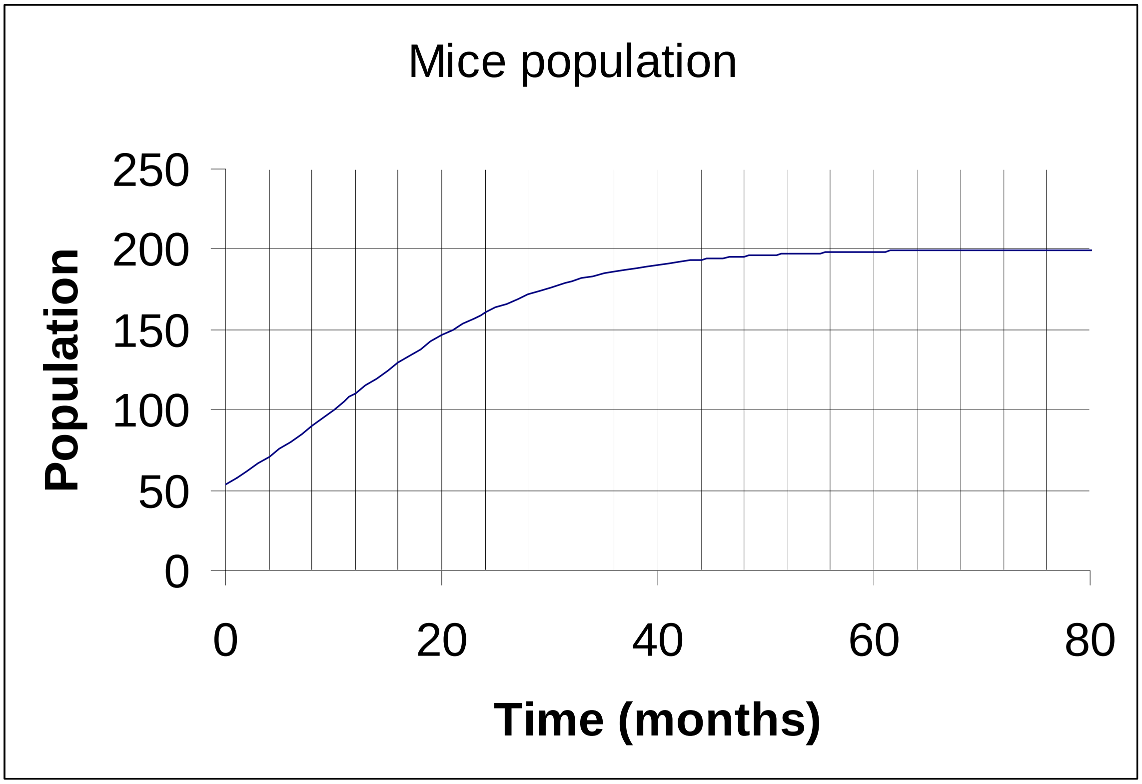
$$\frac{dy}{dt} = y^2(y-8)(y-5) = 0 \Rightarrow y = 0,5,8 \quad Equilibrium \quad points$$

b) Determine whether equilibrium points are stable, unstable (or perhaps something else). **(5 pts)**



9) A graph of mice population is shown below. If the population is described by the Logistics Equation (see the Appendix):

c) Estimate the growth rate k constant. *(Hint: What is 1 + Aexp(-kt) when the initial population doubles?)* **(10 pts)**



a) Determine constants A and K from the graph **(5 pts)**

The plot saturates at 200 mice. This is the carrying capacity, K =200. From the plot the initial population is ~ 50 so that A = (200 – 50)/50 = 3

A = 3, K = 200

b) How long does it take for the initial population to double? **(5 pts)**

From the graph it takes about ~ 11 months.

c) When the initial population doubles: 200 = 100/(1+3exp(-11k))

Rearranging terms: 1 + 3exp(-11k) = 2 or exp(-11k)=1/3

Solving for **k= ln(3)/11 = .01/month**

10) The population of aphids on a rose plant increases at a rate proportional to the number present.

a) Write a differential equation for population of aphids at time t in days. **(10 pts)**

This is an exponential growth equation

$$\frac{dP}{dt} = kP$$

b) Find the solution to the differential equation where at t=0 there were 1000 aphids and population doubles every 10 days. **(10 pts)**

Solution is exponential growth

$$P(t) = P_o \exp(kt)$$
$$2P_o = P_o \exp(10k)$$
$$k = \frac{\ln(2)}{10} = .0693$$
$$P_o = 1000$$

11) Newton's Law of cooling states that the rate of cooling is proportional to the temperature difference between the object and its surrounding. **(10 pts)**

The differential equation for temperature, T, of a coffee cup as a function of time (t) where T_A is the ambient temperature of air, $T_o > T_A$ is the initial temperature of the coffee and k is the proportionality constant is: **(5 pts)**

A) $T' = -kT(1- T/T_A)$
B) $T' = -k(T - T_A)$
C) $T' = k(T - T_A)$
D) $T' = kT/T_A$
E) $T' = -kT/T_A$

Answer is B: Since the coffee is cooling rate must be negative. For $T > T_A$ $T' = -k(T - T_A)$ is always negative and fulfills the description of the Newton's law of cooling.

The solution to the differential equation is: **(5 pts)**

A) $T(t)= T_A/(1+A\exp(-kt))$, where $A = (T_A - T_o)/T_o$
B) $T(t) = T_A + T_o\exp(-kt)$
C) $T(t) = T_A\exp(-kt) - T_A + T_o$
D) $T(t) = T_A + (T_o - T_A)\exp(-kt)$
E) $T(t) = (T_o/T_A)\exp(-kt)$

Answer is D: This is a 1st order linear equation which you can solve using the integration factor method. You can also arrive at this answer by eliminating bad choices.

A) is an answer to Logistics Equation which this is not.
B and E) can be eliminated since at t=0, $T = T_o$. If we set t=0 we get $T_A + T_o$ in B) and T_o/T_A in E)
C doesn't work since for large t, T(t) must approach T_A.

Extra Credit (10 pts)

12) For the following predator pray system determine which of the variables, x or y, represent the prey population and which represent the predator population (Explain). Find equilibrium solutions for preditor and prey.

$$\frac{dx}{dt} = -.05x + .0001xy$$
$$\frac{dy}{dt} = 0.1y - .005xy$$

x must be predator since it depends on y for survival and when y=0, x dies off (x' = -.05x) whereas when x=0, y flourishes (y' = .1y). Prey doesn't need predator.

At equilibrium x'=0 and y' = 0

$$x' = 0 = -.05x + .0001xy \Rightarrow y = 500$$
$$y' = 0 = .1y - .005xy \Rightarrow x = 20$$

Appendix:

Identities:

$$\sec^2(x) = \tan^2(x) + 1$$
$$\cos^2(x) + \sin^2(x) = 1$$
$$\sin^2(x) = \frac{1}{2}\left(1 - \cos(2x)\right)$$
$$\frac{d\tan(\theta)}{d\theta} = \sec^2(\theta)$$
$$\frac{d\sec(\theta)}{d\theta} = \tan(\theta)\sec(\theta)$$

Logistics Equation:

$$\frac{dP}{dt} = kP(K - P)$$
$$P(t) = \frac{K}{1 + A\exp(-kt)}$$
$$A = \frac{(K - P_0)}{P_0}$$

ln(AB)=ln(A) + ln(B)
ln(2)=0.693
ln(3)= 1.10
ln(4)=1.39
ln(5)=1.61