1. (12 points) Answer the following Always True or False. Only your final **boxed** answer will be graded on these problems, and you must write out the words TRUE or FALSE completely.

   (a) We are guaranteed that a unique solution exists for the following initial-value problem for $y(t)$ on the interval $1 \leq t < \infty$: $y'' + \ln(t)y' + y = \ln(t)$, $y(1) = 0$, $y'(1) = 0$.

   (b) Consider the differential equation for $y(t)$: $y'' - y = e^t + t$. $y_p(t) = Ae^t + Bt + C$ is a suitable guess for the particular solution, where A, B, C are constants to be determined.

   (c) $t = -1$ is a *regular singular point* for the following differential equation for $y(t)$:

   $$\frac{d^2y}{dt^2} - \frac{t}{1-t^2}\frac{dy}{dt} + \frac{1}{1-t^2}y = 0$$

   (d) The phase portrait ($y'(t)$ vs. $y(t)$) for the differential equation $y'' - y = 0$ is composed of closed circles centered at the origin.

2. (6 points) Classify, to the best of your abilities, the following differential equations for $y(t)$ (do not solve):

   (a) $t^2\dfrac{d^2y}{dt^2} + 5t\dfrac{dy}{dt} - \cos(t)y = \sinh(t)$

   (b) $y\dfrac{dy}{dt} + y = e^t$

3. (24 points) Solve the following initial value problems for $y(t)$:

   (a) $\dfrac{dy}{dt} = \dfrac{t}{y}, \quad y(4) = -3$

   (b) $\dfrac{dy}{dt} - \dfrac{y}{t} = \left(\dfrac{y}{t}\right)^2, \quad y(1) = 1$ (hint: use transformation $v = y/t$)

   (c) $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} + y = 1, \quad y(0) = y'(0) = 0$

4. (15 points) Consider the following differential equation for $y(t)$:

   $$t^2\frac{d^2y}{dt^2} - y = \frac{1}{t}$$

   (a) Find two linearly independent solutions to the homogeneous problem.

   (b) Show that the two solutions to the homogeneous problem are linearly independent.

   (c) Determine the general solution $y(t)$ for the non-homogeneous problem.

5. (10 points) Consider the system of differential equations $\mathbf{x}' = A\mathbf{x}$ for $\mathbf{x}^T(t) = [x_1(t), x_2(t)]$ where A is defined as

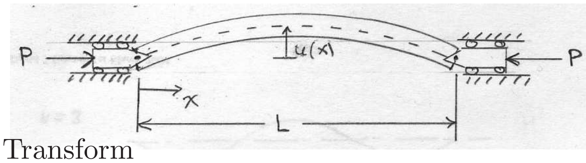
   $$A = \begin{bmatrix} -1 & 2 \\ 4 & 1 \end{bmatrix}.$$

   Determine the general solution and classify the stability of the equilibrium point $\mathbf{x} = \mathbf{0}$.

6. (10 points) Here, we consider a thin elastic beam of length L with simple supports (hinged at both ends) and an axial load P as shown in the figure below. An appropriate mathematical model for small lateral deflection $u(x)$ is

$$\frac{d^2u}{dx^2} + \frac{P}{\alpha}u = 0, \quad u(0) = u(L) = 0,$$

where x is position and α is a constant dependent on beam material and geometry. Clearly, $u(x) = 0$ is a solution. Determine the minimum positive load P for which the beam buckles, i.e., the differential equation has a non-trivial solution.

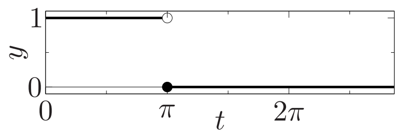


7. (13 points) Laplace Transform

(a) Show that the Laplace transform is a linear operation.

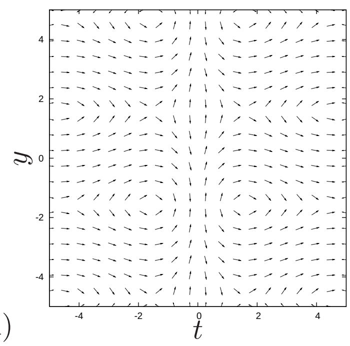
(b) Use the Laplace transform to solve the following initial value problem, where $f(t)$ is plotted below:

$$y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0$$

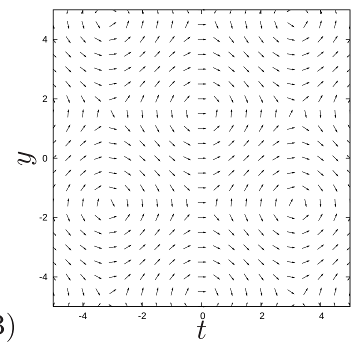


8. (10 points) Match each of the following two ODEs to the appropriate direction field. Only your final answer is graded (A,B,C, or D for each equation).

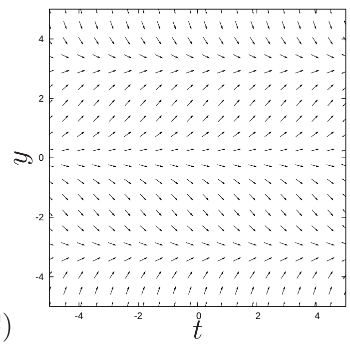
(1) $y' = \frac{\sin(t)}{\cos(y)}$ (2) $y' = y\cos(y/2)$



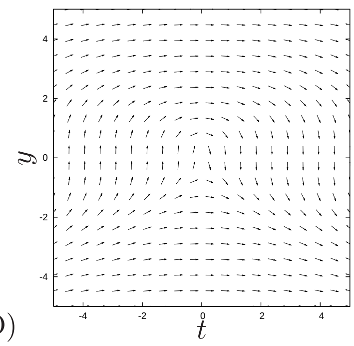
(A)



(B)



(C)



(D)