1. (12 points) Answer the following Always True or False. Only your final boxed answer will be graded on these problems, and you must write out the words TRUE or FALSE completely.

(a) We are guaranteed that a unique solution exists for the following initial-value problem for \( y(t) \) on the interval \( 1 \leq t < \infty \):
\[
y'' + \ln(t)y' + y = \ln(t), \quad y(1) = 0, \quad y'(1) = 0.
\]

(b) Consider the differential equation for \( y(t) \):
\[
y'' - y = e^t + t.
\]
A suitable guess for the particular solution, where \( A, B, C \) are constants to be determined.

(c) \( t = -1 \) is a \textit{regular singular point} for the following differential equation for \( y(t) \):
\[
\frac{d^2y}{dt^2} - \frac{t}{1-t^2} \frac{dy}{dt} + \frac{1}{1-t^2} y = 0
\]

(d) The phase portrait \((y'(t) \text{ vs. } y(t))\) for the differential equation \( y'' - y = 0 \) is composed of closed circles centered at the origin.

2. (6 points) Classify, to the best of your abilities, the following differential equations for \( y(t) \) (do not solve):

(a) \( t^2 \frac{d^2y}{dt^2} + 5t \frac{dy}{dt} - \cos(t)y = \sinh(t) \)

(b) \( y \frac{dy}{dt} + y = e^t \)

3. (24 points) Solve the following initial value problems for \( y(t) \):

(a) \( \frac{dy}{dt} = \frac{t}{y}, \quad y(4) = -3 \)

(b) \( \frac{dy}{dt} - \frac{y}{t} = \left( \frac{y}{t} \right)^2, \quad y(1) = 1 \) \quad (hint: use transformation \( v = y/t \))

(c) \( \frac{d^2y}{dt^2} - 2 \frac{dy}{dt} + y = 1, \quad y(0) = y'(0) = 0 \)

4. (15 points) Consider the following differential equation for \( y(t) \):
\[
t^2 \frac{d^2y}{dt^2} - y = \frac{1}{t}
\]

(a) Find two linearly independent solutions to the homogeneous problem.

(b) Show that the two solutions to the homogeneous problem are linearly independent.

(c) Determine the general solution \( y(t) \) for the non-homogeneous problem.

5. (10 points) Consider the system of differential equations \( \mathbf{x}' = A\mathbf{x} \) for \( \mathbf{x}^T(t) = [x_1(t), x_2(t)] \) where \( A \) is defined as
\[
A = \begin{bmatrix} -1 & 2 \\ 4 & 1 \end{bmatrix}.
\]
Determine the general solution and classify the stability of the equilibrium point \( \mathbf{x} = 0 \).
6. (10 points) Here, we consider a thin elastic beam of length \( L \) with simple supports (hinged at both ends) and an axial load \( P \) as shown in the figure below. An appropriate mathematical model for small lateral deflection \( u(x) \) is

\[
\frac{d^2 u}{dx^2} + \frac{P}{\alpha} u = 0, \quad u(0) = u(L) = 0,
\]

where \( x \) is position and \( \alpha \) is a constant dependent on beam material and geometry. Clearly, \( u(x) = 0 \) is a solution. Determine the minimum positive load \( P \) for which the beam buckles, i.e., the differential equation has a non-trivial solution.

7. (13 points) Laplace Transform

(a) Show that the Laplace transform is a linear operation.

(b) Use the Laplace transform to solve the following initial value problem, where \( f(t) \) is plotted below:

\[
y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0
\]

8. (10 points) Match each of the following two ODEs to the appropriate direction field. Only your final answer is graded (A,B,C, or D for each equation).

(1) \( y' = \frac{\sin(t)}{\cos(y)} \)  \quad (2) \( y' = y \cos(y/2) \)