1. (9 points - 3 points each) Answer each of the following:

   (a) Suppose you happen to know that

   $$(m-1)(m-2)(m-3)(m-4) = m^4 - 10m^3 + 35m^2 - 50m + 24.$$

   Find the general solution of $y'''' - 10y''' + 35y'' - 50y' + 24y = 0$.

   (b) For the differential equation $\dfrac{dy}{dt} = \exp(y^2) - 1$, find the equilibrium solution and determine its stability.

   (c) Sketch the phase-plane ($y'(t)$ vs. $y(t)$) trajectory for the differential equation $y'' - 4y = 0$ with $y(0) = 1$ and $y'(0) = 1$.

2. (18 points - 6 points each) Find the solutions to the following initial-value problems:

   (a) $x^2 \dfrac{dy}{dx} = xy - y^2, \quad y(1) = 1, \quad$ hint: use transformation $y = ux$

   (b) $\dfrac{dy}{dx} + (\tan x)\, y = \sec x, \quad y(0) = 4$

   (c) $\dfrac{x}{y^2}\dfrac{dy}{dx} = 1 + \dfrac{1}{y}, \quad y(1) = \dfrac{1}{2}$

3. (10 points) Consider the first-order equation

   $$\frac{dx}{dt} + x = f(t).$$

   (a) Solve the equation for $x(t)$, assuming only that $f(t)$ is continuous.

   (b) Find a function $f(t)$ such that $f(t) > 0$ for all t, *and* regardless of the initial condition $x(0)$, all solutions $x(t)$ of the first-order equation satisfy

   $$\lim_{t\to\infty} x(t) = 0.$$

   Be sure to include relevant details that explain why your function $f(t)$ works.

4. (18 points) Consider the equation

   $$y'' + y = \sin(\alpha x), \tag{1}$$

   where α is a positive real number.

   (a) Write down two linearly independent solutions to the homogeneous problem. How do you know they are linearly independent?

   (b) Assuming $\alpha \neq 1$, find the general solution $y_\alpha(x)$ of (1).

   (c) Return to (1), set $\alpha = 1$, and find the general solution $y_1(x)$.

   (d) True/False: in the $\alpha \to 1$ limit, $y_\alpha(x)$ approaches $y_1(x)$.

5. (15 points) Consider the boundary-value problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(\pi) = 0.$$

(a) Find the eigenvalues λ_n and the linearly independent eigenfunctions $y_n(x)$. Number the eigenvalues/eigenfunctions so that the smallest eigenvalue corresponds to $n = 0$.

(b) Sketch the first four eigenfunctions for $x \in [0, \pi]$.

(c) Consider the function

$$f(x) = \begin{cases} 1 & 0 \le x < \pi/2 \\ -1 & \pi/2 \le x \le \pi. \end{cases}$$

Show that if n is even,

$$\int_0^\pi f(x) y_n(x)\, dx = 0.$$

Relate this result to symmetries and/or antisymmetries of $f(x)$ and $y_n(x)$.

6. (15 points) Consider the equation

$$\frac{d^2x}{dt^2} + x - x^3 = 0. \tag{2}$$

(a) Rewrite the second-order equation (2) as a first-order system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ f(x) \end{pmatrix},$$

where $f(x)$ is a function you must determine.

(b) Linearize the first-order system about the equilibrium solutions $(\pm 1, 0)$. Your result should be a 2×2 matrix A.

(c) From the linearization, determine the stability of the equilibrium solutions $(\pm 1, 0)$.

(d) How, if at all, do the results of the previous parts change if the $-x^3$ term in (2) is replaced by $-x^{2n+1}$ where n is any positive integer?

7. (15 points - 5 points each) For each choice of the matrix A, find the general solution of

$$\frac{d}{dt}\mathbf{v}(t) = A\mathbf{v}, \ \ \mathbf{v}(0) = \mathbf{z}.$$

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.