Directions: This examination lasts 4 hours. Books and notes are allowed, but no electronic device of any kind is permitted. Partial credit will be awarded to any relevant work; no credit will be awarded for unexplained answers, correct or not. Computations mistakes will be very lightly penalized. Accurate graphic representation of a problem will receive high consideration.

1. Find all complex numbers z such that the following equalities hold and indicate where these points are on the complex plane.
   (a) $z^3 = i$
   (b) $\log z = 1 + \frac{i\pi}{4}$
   (c) $z = \log 2i$
   (d) $\bar{\bar{z}} = \bar{z}$
2. Using Euler's formula, derive the formulæ for $\sin(a+b)$ and $\cos(a+b)$ in terms of $\sin a$, $\sin b \cos a$ and $\cos b$, where $a, b \in \mathbb{R}$.
3. If $z = x + iy$ and $f(z) = x^2$, determine where $f'(z)$ exists and what its value is.
4. Determine the Taylor Series centered at $z = 2i$ of $f(z) = \frac{1}{z}$ and determine its radius of convergence. From the Taylor Series, find $f^{(4)}(2i)$.
5. Evaluate $\int_C z^{n-1} f(z) dz$ where n is any positive integer, C is the octagon centered x = 0.2 with sides of length 20 and
$$f(z) = \frac{1}{z+1} + e^{\frac{i}{z}} + \sinh(e^{z^3+1})$$
6. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2-2x+2)^2}$ using Cauchy's Residue Theorem.
7. Evaluate $\int_0^{\infty} \frac{dx}{x^{2/3}(x^2+1)}$ using Cauchy's Residue Theorem.
8. Find the inverse Laplace transform of $F(s) = \frac{1}{s \sinh s}$, in terms of the residues of some complex function. Identify where the residues have to be evaluated and the order of the corresponding poles.
9. Describe in words and by mapping relevant domains the following transforms of the z-plane onto the w-plane.
   (a) $w = f(z) = iz^3$
   (b) $w = f(z) = \frac{4}{z}$

10. Using a logarithmic mapping, find the steady temperature distribution (which is by definition a harmonic function) in the upper half-plane portion of the exterior of the disk of radius one such that

$$T(r,\theta) = 1 \quad \text{if} \quad \theta = 0 \quad \text{and} \quad r > 1 \tag{1}$$
$$T(r,\theta) = 1 + \pi\theta \quad \text{if} \quad r = 1 \tag{2}$$
$$T(r,\theta) = 1 + \pi^2 \quad \text{if} \quad \theta = \pi \quad \text{and} \quad r > 1. \tag{3}$$

Express your solution in cartesian coordinates, i.e. find T(x,y).

11. Consider a simple closed curve C parametrized by $z_C(t)$, with $t \in [0,1]$ and let the interior of C be the domain D_z. Assume that you are given an invertible function $h(z)$ which maps D_z onto the disk of unit radius centered at the origin.

a) Draw the a representation of the mapping and label as many components as possible.

b) Find an expression for the harmonic function $f(z)$ satisfying $\nabla^2 f = 0$ over D_z and subject to the boundary condition $f(z_C(t)) = g(t)$.