University of California Merced

Capstone Project

Governing Equations For Simulations Of Soap Bubbles

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1 Introduction

The capstone project is an extension of work previously done by *François Blanchette* and *Terry P. Bigioni* in which they studied the coalescence of drops with either a horizontal reservoir or a drop of a different size (Blanchette and Bigioni, 2006). Specifically, they looked at the partial coalescence of a drop, which leaves behind a smaller "daughter" droplet due to the incomplete merging process. Numerical simulations were used to study coalescence of a drop slowly coming into contact with a horizontal resevoir in which the fluid in the drop is the same fluid as that below the interface (Blanchette and Bigioni, 2006).

The research conducted by Blanchette and Bigioni starts with a drop at rest on a flat interface. A drop of water will then merge with an underlying resevoir (water in this case), forming a single interface. Our research involves using that same numerical approach only this time a soap bubble will be our fluid of interest.

Soap bubbles differ from water drops on the fact that rather than having just a single interface, we now have two interfaces to take into account; the air inside the soap bubble along with the soap film on the boundary, and the soap film with any other fluid on the outside. Due to this double interface, some modifications will now be imposed on the boundary conditions involving surface tension along the interface. Also unlike drops, soap bubble thickness is finite which will mean we must keep track of it. As a consequence, the film will have a weight and therefore a new forcing term will be added to the governing equations. There are two things that are hoped to be achieved by some numerical simulations; one is to study the interaction of two soap bubbles, or of a soap bubble and a soap film. The second is to study the interaction of a soap film and some air flow.

2 Governing Equations

In the research done by Blanchette and Bigioni, the equations used were those of the Navier-Stokes (1). The fluids were assumed to be ideal, meaning that they are incompressible (2), with constant density ρ and viscosity μ. The velocity at the interface location must be the same as the time derivative of the position interface (3):

$$\rho \frac{D\vec{u}}{Dt} = -\nabla P + \mu \nabla^2 \vec{u} + \rho \vec{g} \tag{1}$$

$$\nabla \cdot \vec{u} = 0 \tag{2}$$

$$S_t = \vec{u} \tag{3}$$

In the equations above, $\vec{u}$ is the velocity field of the fluid; P is the pressure field; t represents the time; S is the position of the interface; $\vec{g}$ is a vector in the direction of gravity; $\frac{D}{Dt}$ is the material derivative that tracks the position of the fluid particle.

2.1 Boundary Conditions

Along with the governing equations of the fluid, boundary conditions must also be imposed and satisfied. The boundary conditions are as follows:

$$\vec{n} \cdot T - \vec{n} \cdot T^{'} = \sigma \vec{n}(\nabla \cdot \vec{n}) - \nabla \sigma \tag{4}$$

where $T = -PI + \mu[\nabla \vec{u} + (\nabla \vec{u})^T]$ is the inner fluid stress tensor defined in terms of the velocity field as well as the local fluid pressure; $T^{'}$ defined as the same stress tensor but it serves for the outside fluid; $\vec{t}$ is the unit vector tangential to the interface; σ is the surface tension; and $\vec{n}$ is a normal unit vector that is normal to the interface and going in the direction of the inner fluid towards the outer fluid. Condition (4) is at the interface, and is the *Stress Balance* equation that allows the difference in stress vectors for both the inner and outer fluid to be the same as the difference between the normal curvature force and the tangential stress involving gradients of surface tension (Bush, 2003). There is also the no-slip condition:

$$\vec{u} = 0 \tag{5}$$

Boundary condition (5) acts only on the side-walls due to axial symmetry in order for the fluid to be finite. Numerically, it's very difficult to deal with the boundary conditions (4) on their own. To fix this, we will include a forcing term $\delta_s[\sigma \vec{n}(\nabla \cdot \vec{n}) - \nabla \sigma]$ in the Navier Stokes equation (1) (Lafaurie *et al.* 1994), which takes into account the boundary conditions. In the forcing term, δ_s is the surface Dirac delta function which is used in order to limit the term to be non-zero only at the interface, and zero everywhere else; and as result it yields the following:

$$\rho\frac{D\vec{u}}{Dt} = -\nabla P + \mu\nabla^2\vec{u} + \rho\vec{g} + \delta_s[\sigma\vec{n}(\nabla\cdot\vec{n}) - \nabla\sigma] \tag{6}$$

$$\nabla\cdot\vec{u} = 0 \tag{7}$$

$$S_t = \vec{u} \tag{8}$$

In fact, we can say that this additional term is equivalent to the boundary conditions through an extension of *Duhamel's principle*. It will be easier to use the boundary conditions numerically rather than to match stresses as in equation (4).

3 Modification of the Governing equations

3.1 Derivation Of The Soap Film Thickness

As metioned before, since the thickness of soap bubble is finite, the soap film will now have a weight associated with it. The weight of the interface will be a force driven by gravity. The weight of the interface is given by $m\vec{g}$ where m is the localized mass of the soap film interface. The mass can also be related by both its density and volume, and therefore can be written as $\rho dV\vec{g}$ where dV is a volume element. We can also relate the volume of the inteface to its thickness as the following $dV = hdA$ where $h = h(x, y, t)$ represents the thickness of the soap film and $dA = \Delta x\Delta y$ represents the area element of the interface. The main issue is going to be keeping track of how the thichkness of the soap film is changing as time evolves, and so we will have to look at the term $\frac{\partial h}{\partial t}$. Let's consider a simple case where the have a flat surface in the $xy-$plane on the soap bubble. The rate of change of the thickness can be described as $\frac{\partial h}{\partial t} = \frac{\textit{Flow Rate in - Flow Rate out}}{\textit{Area}}$, and so this yields the following expression:

$$\frac{\partial h}{\partial t} = -\lim_{\Delta x \to 0, \Delta y \to 0} \sum_{i=1}^{4} \frac{1}{\Delta x \Delta y} (\vec{u} \cdot \vec{n_i}) dA_i \tag{9}$$

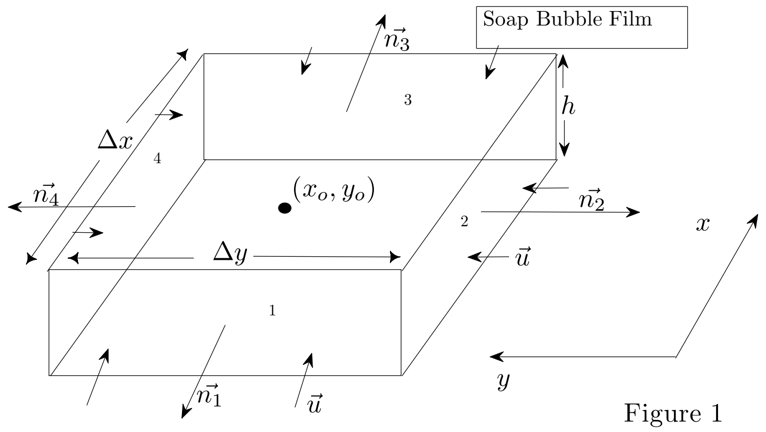

Figure 1

The dA_i terms represents the area of the four faces of the soap film, n_i are the unit normals of the each face respectively where $i = 1, 2, 3, 4$. The $\vec{u} \cdot \vec{n_i} dA_i$ terms each represent the flow along a particular face (**See Figure 1**). The normals of each face are as follows: $\vec{n_1} =< -1, 0 >$, $\vec{n_2} =< 0, -1 >$, $\vec{n_3} = -\vec{n_1}$, and finally $\vec{n_4} = -\vec{n_2}$. The areas of the four elements are as follows: $dA_1 = h\Delta y = dA_3$ and $dA_2 = h\Delta x = dA_4$. By letting the velocity field $\vec{u}(x, y, t) =< u_1, u_2 >$ where $u_1 = u_1(x, y, t)$ and $u_2 = u_2(x, y, t)$ and taking the all dot products for faces 1 and 3 on (9) yields the following:

$$\frac{u_1(x_0 + \frac{\Delta x}{2}, y_0, t)h(x_0 + \frac{\Delta x}{2}, y_0, t) - u_1(x_0 - \frac{\Delta x}{2}, y_0, t)h(x_0 - \frac{\Delta x}{2}, y_0, t)}{\Delta x}$$

repeating the same process but this time on faces 2 and 4 would yield a similar expression. At the end, after taking all the dot products and evaluating the limit on equation (9), we obtain the desired result:

$$\frac{\partial h}{\partial t} = -[\frac{\partial}{\partial x}(u_1(x_0,y_0,t)h(x_0,y_0,t)) + \frac{\partial}{\partial y}(u_2(x_0,y_0,t)h(x_0,y_0,t))] = -\nabla \cdot (h\vec{u}) \qquad (10)$$

This formula represents the change in thickness for a two dimensional flat surface. In general, for any surface the change in thickness of any soap film is as follows:

$$\frac{\partial h}{\partial t} = -[\frac{\partial}{\partial \tilde{x}}(h\vec{u}\cdot\vec{e_1}) + \frac{\partial}{\partial \tilde{y}}(h\vec{u}\cdot\vec{e_2})] = -\nabla_s \cdot (h\vec{u}). \qquad (11)$$

where ∇_s is the surface gradient, $\nabla_s \cdot (h\vec{u})$ is the surface divergence of the velocity $\vec{u}(\tilde{x},\tilde{y},t) =< u_1(\tilde{x},\tilde{y},t), u_2(\tilde{x},\tilde{y},t) >$ on any given surface in the $\tilde{x}\tilde{y}$ orthogonal axis where both $\vec{e_1}$,$\vec{e_2}$ are the orthonormal basis.

3.2 Derivation Of The Gravitational Forcing Term

Now there is the matter of including this forcing term into the Navier Stokes equation. The forcing term is actually the weight of the interface, the force driven by gravity. This term will be of the form $\frac{\text{Force}}{\text{Volume}}$. This forcing term is known as the **total gravitational force**, i.e total gravitational force = $\int_V \vec{f_g} dV = \int_S \rho\vec{g}h dS$. The question is what exactly is $\vec{f_g}$? From this expression, it's definitely true to say that $\vec{f_g} = \rho\vec{g}h\frac{dA}{dV}$ where dA is the area of the film and dV is the volume over which the force acts. This however is not a useful term since both dA and dV are not local quantities. Looking back at the total gravitational expression, the surface delta function would be more useful since it's a term that acts locally on the surface, and it has units of $\frac{1}{L}$, which is exactly the same units the expression $\frac{area}{volume}$ would give. Keeping this idea in mind, this total gravitational force can we written as: $\int_V \vec{f}_g dV = \int_V \rho\vec{g}h\delta_s dV = \int_S \rho\vec{g}hdS$, and thus gives the desired forcing term for the Navier Stokes equation: $\vec{f}_g = \rho\vec{g}h\delta_s$. Also soap bubble in air have two surfaces that define the inner and outer surfaces of the soap film; as a consequnce, the pressure differntial term is twice that across of a single interface, i.e $\Delta p = 2[\sigma\vec{n}(\nabla\cdot\vec{n}) - \nabla\sigma]$ (Bush, 2003). Finally, we now have the modified governing equations of the soap bubble film as well as the soap film thickness as follows:

$$\rho\frac{D\vec{u}}{Dt} = -\nabla P + \mu\nabla^2\vec{u} + \rho\vec{g} + \delta_s[2(\sigma\vec{n}(\nabla\cdot\vec{n}) - \nabla\sigma) + \rho\vec{g}h] \tag{12}$$

$$\nabla\cdot\vec{u} = 0 \tag{13}$$

$$S_t = \vec{u} \tag{14}$$

$$\frac{\partial h}{\partial t} = -\nabla_s\cdot(h\vec{u}). \tag{15}$$

4 Numerical Simulations

There is a numerical simulation code called **SURFER** used by Blanchette and Bigioni to study the partial coalescence of water drops. One can also adopt the same program to study the interactions between two soap bubbles merging or the interaction with another fluid like air. The adaptation of the new forcing term in of the modified Navier Stokes equations in **SURFER** shouldn't be more difficult since the boundary condition term has already been dealt with previously, and h will be a given quantity due to equation (11). Numerically, equation (11) can be solved by using a centered finite difference method for both space and time. Also numerically the main issue is being able to calculate the surface divergence since the interface may not always be horizontal. Assume that, just as in the current code, the interface is a surface of revolution of a curve given by a radius $r(z)$ where the z axis is the axis of rotation. Then allowing $x = r(z)sin\theta$, $y = r(z)cos\theta$, and $z = z$ where θ is angle of rotation, the surface of revolution can be represented parametrically. Afterwards, the surface gradient will be defined on a orthogonal system which would allow a surface divergence to be calculated. An example would be taking a bubble in the shape of a sphere. The curve will be of the form $r(z) = \sqrt{1 - z^2}$ rotated about the z axis. One will then represent the surface gradient and the velocity in spherical coordinates on a orthogonal system in order to calculate a surface divergence.

5 Conclusion

The thickness of the soap film being finite lead to the idea of finding the expression that could keep track of how the the film changes with respect to time; because of this, it also yield the gravitational forcing term which was added into the Navier Stokes equations for numerical simulations. Further research can be done on **SURFER**, by modifying the current program with equation (12-15) to study the coalescence between two soap bubbles or a soap bubble with any other fluid. For example, as Blanchette and Bigioni did partial coalescence of water drops, perhaps a smiliar phenomenon could occur with soap bubble, where possibly a daughter bubble can form after the merging with another bubble. Another application would be the study of bubbles bouncing back from a surface like bath water, or the possibility of bubble deformation.

References

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