Duration: 4 hours.

Instructions:

- Answer all the questions. You may use one cheat-sheet.
- Partial credit will be awarded for correct relevant work.
- No credit will be awarded for unexplained answers, correct or not.
- Computational mistakes will be very lightly penalized.
- Where relevant, accurate graphic representations will receive high consideration.
- All questions are worth the same number of points.

1. Find the real and imaginary parts of **all the complex solutions** of the equations below.
   (a) $z^3 = 1 - i$.
   (b) $e^z = 1 + i$.
   (c) $\tan z = 1 + i$.
   (d) $\cosh^2 z - \sinh^2 z = 1$.

2. Prove Lagrange's identity:

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta \;=\; \frac{1}{2} + \frac{\sin\left[\frac{(2n+1)\theta}{2}\right]}{2\sin\left(\frac{\theta}{2}\right)}$$

   Hint: consider the sum $S = 1 + z + \ldots z^n$ with $z = e^{i\theta}$.

3. Let the derivative operator with respect to $\bar{z}$ be defined as

$$\frac{\partial}{\partial \bar{z}} \doteq \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) .$$

   (a) Let $f(z)$ be analytic. Prove that $\dfrac{\partial f(z)}{\partial \bar{z}} = 0$.
   (b) Find $\dfrac{\partial(|z|^2)}{\partial \bar{z}}$.

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4. Compute $\int_C |z|^2 \, dz$ over the semi-circle in the upper half plane from -1 to 1.

5. Compute the Laurent series of $\dfrac{1}{2+z-z^2}$ that is valid in the region $1 < |z| < 2$.

6. Compute the residues at **all the isolated singularities** of the functions

   (a) $\dfrac{\operatorname{Log}(z)}{(z^2+1)^2}$,

   (b) $\dfrac{1}{\sin z}$.

7. Use Cauchy's Residue Theorem to evaluate

   (a) $\displaystyle\int_0^\infty \frac{e^{-x}}{\sqrt{x}} \, dx$.

   (b) $\displaystyle\int_{-\infty}^\infty \frac{e^{itx}}{1+x^2} \, dx$, where t is a positive constant.

8. (a) Sketch what the map $w = \dfrac{z+i}{z-i}$ does to:

      i. the upper half plane $y \geq 0$,

      ii. the line $\operatorname{Im}[z] = 0$,

      iii. the circle $|z| = 1$.

   (b) Sketch what the map $w = \sin z$ does to the strip $\left\{ |x| \leq \dfrac{\pi}{2}, \ y \geq 0 \right\}$ and its boundaries.

9. Find a function $f(z)$ that is analytic in the right half plane $x > 0$, such that $\operatorname{Re}[f(z)] = \ln \sqrt{x^2+y^2}$.

10. Use the conformal map $w(z) = \operatorname{Log}\left(\dfrac{z-1}{z+1}\right)$ to find a harmonic function $T(x,y)$ that satisfies the boundary value problem

$$
\begin{aligned}
T_{xx} + T_{yy} &= 0 \quad \text{in} \quad y > 0 \ , \\
T &= 0 \quad \text{on} \quad |x| < 1, \ y = 0 \ , \\
T &= 1 \quad \text{on} \quad |x| > 1, \ y = 0 \ .
\end{aligned}
$$

THE END