Instructions:

- Answer all the questions. You may use one cheat-sheet.
- Partial credit will be awarded for correct relevant work.
- No credit will be awarded for unexplained answers, correct or not.
- Computational mistakes will be very lightly penalized.
- Where relevant, accurate graphic representations will receive high consideration.
- All questions are worth the same number of points.

1. Find all the complex numbers $z$ such that the following equalities hold and indicate where these points are on the complex plane.
   
   (a) $z^3 = -i$
   
   (b) $e^z = \sqrt{i}$
   
   (c) $\bar{z} = z^3$
   
   (d) $\arccos z + \arcsin z = \frac{\pi}{2}$

2. Prove the identity
   
   $$\text{Im} \left( \frac{1 + re^{i\theta}}{1 - e^{i\theta}} \right) = \frac{2r \sin \theta}{1 - 2r \cos \theta + r^2}.$$ 

3. Let $\Gamma(z)$ be defined as
   
   $$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}, \quad z \neq 0, -1, -2, \ldots$$

   where $\gamma$ is a constant. Shew that
   
   $$\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma - \sum_{n=1}^{\infty} \left( \frac{1}{z + n} - \frac{1}{n} \right).$$

   (Hint: consider the integral of the the bottom expression).
4. Compute \[ \int_C \frac{z + 2}{z} \, dz \] over the semi-circle \( z = 2e^{i\theta}, \ \theta \in [0, \pi] \).

5. Compute the Laurent series of \( \frac{1}{(z - 1)(z - 2)} \) that is valid in the region \( 1 < |z| < 2 \).

6. Compute the residues at all the isolated singularities of the functions
   (a) \( \frac{z}{\sin^2 z} \)
   (b) \( \frac{(\log z)^2}{z^2 + 1} \)

7. Use Cauchy’s Residue Theorem to evaluate
   (a) \( \int_0^\infty \frac{dx}{1 + x^4} \).
   (b) \( \int_{-\infty}^\infty \frac{\sin x}{x} \, dx \).

8. (a) Sketch what the map \( w = \log \frac{z - 1}{z + 1} \) does to:
   i. the upper half plane \( \text{Im}[z] \geq 0 \)
   ii. the line \( \text{Im}[z] = 0 \)

9. Find a function \( f(z) \) that is analytic in the right half plane \( x > 0 \), such that \( \text{Re}[f(z)] = \arctan(y/x) \).

10. Find a harmonic function \( T(x, y) \) in the half space \( y > 0 \), such that \( T = 1 \) on \( |y| \leq a \) and \( T = 0 \) on \( |y| > a \) for some constant \( a > 0 \). Verify that this function satisfies the boundary conditions.