Duration: 4 hours.

Instructions:

- Answer all the questions. You may use one cheat-sheet.
- Partial credit will be awarded for correct relevant work.
- No credit will be awarded for unexplained answers, correct or not.
- Computational mistakes will be very lightly penalized.
- Where relevant, accurate graphic representations will receive high consideration.
- All questions are worth the same number of points.

1. Find **all the complex numbers** z such that the following equalities hold and indicate where these points are on the complex plane.

   (a) $z^3 = -i$

   (b) $e^z = \sqrt{i}$

   (c) $\bar{z} = z^3$

   (d) $\arccos z + \arcsin z = \dfrac{\pi}{2}$

2. Prove the identity

$$\operatorname{Im}\left(\frac{1+re^{i\theta}}{1-e^{i\theta}}\right) \;=\; \frac{2r\sin\theta}{1-2r\cos\theta+r^2}\ .$$

3. Let $\Gamma(z)$ be defined as

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}\ , \quad z \neq 0, -1, -2, \ldots$$

   where γ is a constant. Shew that

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma - \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right)\ .$$

   (Hint: consider the integral of the the bottom expression).

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4. Compute $\int_C \frac{z+2}{z}\,dz$ over the semi-circle $z = 2e^{i\theta},\ \theta \in [0, \pi]$.

5. Compute the Laurent series of $\frac{1}{(z-1)(z-2)}$ that is valid in the region $1 < |z| < 2$.

6. Compute the residues at **all the isolated singularities** of the functions

   (a) $\frac{z}{\sin^2 z}$

   (b) $\frac{(\operatorname{Log} z)^2}{z^2+1}$

7. Use Cauchy's Residue Theorem to evaluate

   (a) $\int_0^{\infty} \frac{dx}{1+x^4}$.

   (b) $\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$.

8. (a) Sketch what the map $w = \operatorname{Log} \frac{z-1}{z+1}$ does to:

   i. the upper half plane $\operatorname{Im}[z] \geq 0$

   ii. the line $\operatorname{Im}[z] = 0$

9. Find a function $f(z)$ that is analytic in the right half plane $x > 0$, such that $\operatorname{Re}[f(z)] = \arctan(y/x)$.

10. Find a harmonic function $T(x, y)$ in the half space $y > 0$, such that $T = 1$ on $|y| \leq a$ and $T = 0$ on $|y| > a$ for some constant $a > 0$. Verify that this function satisfies the boundary conditions.

THE END