1. (15 pts) Plot all complex numbers satisfying the following equalities or inequalities.
   a) \(z^3 = -9i\)
   b) \(|2z - 1 - i| \leq 2\).
   c) \(z = (1 + i)^i\)

2. (21 pts) In the following proof of the Cauchy Integral formula, explain why each of the seven numbered statements hold. Your explanations should be brief (one sentence or formula for each statement).

Thm: If \(f(z)\) is analytic on and inside the closed contour \(C\) and \(z_0\) is inside \(C\), then
\[
\frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz - f(z_0) = 0.
\]

Proof: Let \(C_\rho\) be a circle of radius \(\rho\) centered at \(z_0\), with \(\rho\) sufficiently small that \(C_\rho\) is entirely inside \(C\).
\[
\frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz - f(z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z - z_0} dz - f(z_0) \tag{1}
\]
\[
\frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z - z_0} dz - f(z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz \tag{2}
\]
Also, \(\forall \epsilon > 0, \exists \rho > 0\) such that \(|z - z_0| \leq \rho \rightarrow |f(z) - f(z_0)| < \epsilon\), so
\[
\left|\frac{1}{2\pi i} \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz\right| \leq \frac{1}{2\pi} \int_{C_\rho} \left|\frac{f(z) - f(z_0)}{z - z_0}\right| |dz| \tag{4}
\]
\[
\frac{1}{2\pi} \int_{C_\rho} \left|\frac{f(z) - f(z_0)}{z - z_0}\right| |dz| < \frac{1}{2\pi} \int_{C_\rho} \frac{\epsilon}{\rho} |dz| \tag{5}
\]
\[
\frac{1}{2\pi} \int_{C_\rho} \frac{\epsilon}{\rho} |dz| \leq \epsilon \tag{6}
\]
Therefore \(\frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz - f(z_0) = 0\). \(\tag{7}\)

3. (20 pts) Find the general form of the requested series and give its radius of convergence
   a) Find the Taylor series of \(f(z) = \frac{z - 1}{z + 2i}\) centered at \(z = 1\).
   b) Find the Laurent Series centered at \(z = 0\) of \(f(z) = \sin\left(\frac{1}{z}\right)\)

4. (15 pts) Evaluate \(\int_{-\infty}^{\infty} \frac{1}{(x^2 + 10)^2} dx\)
5. Evaluate \( \int_0^\infty \frac{x^{1/4}}{x^2+9} \, dx \)

6. Consider the function \( G(z) = \frac{\cos(\pi z)}{z^2 \sin(\pi z)} \). You may use the identities:
   \[ \cos(z) = \cos x \cosh y - i \sin x \sinh y \]
   \[ \sin(z) = \sin x \cosh y + i \cos x \sinh y. \]
   
   a) Find and classify all the singularities of \( G(z) \).
   
   b) Find the residues at all the singularities of \( G(z) \) (you may use the table provided at the end of the exam).
   
   c) Consider the contour \( S \) made of a square centered at the origin, extending from \( -(n + 1/2) \) to \( (n + 1/2) \), with \( n \in \mathbb{N} \), in both the \( x \) and \( y \) direction. Bound \( \int_S G(z) \, dz \) as a function of \( n \).
   
   d) Take the limit as \( n \to \infty \) and use the Residue theorem, to obtain a famous identity.

7. The function
   \[ f(z) = \sum_{n=2}^{\infty} \frac{n \, i^n \, z^n}{(n - 1)!} \]
   is analytic and entire. Using integration or derivation of known Taylor Series, and multiplication/addition/subtraction of polynomials, find a closed formed expression for \( f(z) \).

8. Using a logarithmic mapping, find the steady state temperature (satisfying \( \nabla^2 T = 0 \)) in the region between the lines \( y = x \) and \( y = \sqrt{3} \, x \), in the first quadrant, if the temperature is \( T = 2 \) along \( y = x \) and \( T = 3 \) along \( y = \sqrt{3} \, x \).

9. Find the mapping required to transform the quarter-circle centered at \( 1 - 2i \) going from \( 3 - 2i \) to \( 1 \), into the lower half-circle of radius 1 centered at the origin.
\[
\frac{d}{dz} \left( \frac{\cos(\pi z)}{z \sin(\pi z)} \right) = -(\cot(\pi z)/z^2) - (\pi \csc^2(\pi z))/z
\]

\[
\frac{d^2}{dz^2} \left( \frac{\cos(\pi z)}{z \sin(\pi z)} \right) = 2 \cot(\pi z)/z^3 + (2 \pi \csc^2(\pi z))/z^2 + (2 \pi^2 \cot(\pi z) \csc^2(\pi z))/z
\]

\[
\frac{d^3}{dz^3} \left( \frac{\cos(\pi z)}{z \sin(\pi z)} \right) = (6 \cot(\pi z))/z^4 - (6 \pi \csc^2(\pi z))/z^3 - (6 \pi^2 \cot(\pi z) \csc^2(\pi z))/z^2 - (4 \pi^3 \cot(\pi z)^2 \csc^2(\pi z))/z - (2 \pi^3 \csc^4(\pi z))/z
\]

\[
\frac{d}{dz} \left( \frac{\cos(\pi z)}{\sin(\pi z)} \right) = -\left( \pi \csc^2(\pi z) \right)
\]

\[
\frac{d^2}{dz^2} \left( \frac{\cos(\pi z)}{\sin(\pi z)} \right) = 2 \pi^2 \cot(\pi z) \csc^2(\pi z)
\]

\[
\frac{d^3}{dz^3} \left( \frac{\cos(\pi z)}{\sin(\pi z)} \right) = -4 \pi^3 \cot^2(\pi z) \csc^2(\pi z) - 2 \pi^3 \csc^4(\pi z)
\]

\[
\frac{d}{dz} \left( \frac{z \cos(\pi z)}{\sin(\pi z)} \right) = \cot(\pi z) - \pi z \csc^2(\pi z)
\]

\[
\frac{d^2}{dz^2} \left( \frac{z \cos(\pi z)}{\sin(\pi z)} \right) = -2 \pi \csc^2(\pi z) + 2 \pi^2 z \cot(\pi z) \csc^2(\pi z)
\]

\[
\frac{d^3}{dz^3} \left( \frac{z \cos(\pi z)}{\sin(\pi z)} \right) = 6 \pi^2 \cot(\pi z) \csc^2(\pi z) - 4 \pi^3 z \cot^2(\pi z) \csc^2(\pi z) - 2 \pi^3 z \csc^4(\pi z)
\]

\[
\frac{d}{dz} \left( \frac{z^2 \cos(\pi z)}{\sin(\pi z)} \right) = 2 z \cot(\pi z) - \pi z^2 \csc^2(\pi z)
\]

\[
\frac{d^2}{dz^2} \left( \frac{z^2 \cos(\pi z)}{\sin(\pi z)} \right) = 2 \cot(\pi z) - 4 \pi z \csc^2(\pi z) + 2 \pi^2 z^2 \cot(\pi z) \csc^2(\pi z)
\]

\[
\frac{d^3}{dz^3} \left( \frac{z^2 \cos(\pi z)}{\sin(\pi z)} \right) = -6 \pi \csc^2(\pi z) + 12 \pi^2 z \cot(\pi z) \csc^2(\pi z) - 4 \pi^3 z^2 \cot^2(\pi z) \csc(\pi z) - 2 \pi^3 z^2 \csc^4(\pi z)
\]