Duration: 240 minutes

Answer all questions. Partial credit will be awarded to correct but partial work. Points will be deducted for non-sensical answers. This test is meant to be difficult, so you are not expected to be able to answer every question perfectly.

1. (15 pts) **Plot** all complex numbers satisfying the following equalities or inequalities.

   a) $z^3 = -9i$

   b) $|2z - 1 - i| \leq 2$.

   c) $z = (1 + i)^i$

2. (21 pts) In the following proof of the Cauchy Integral formula, explain why each of the seven numbered statements hold. Your explanations should be brief (one sentence or formula for each statement).

   Thm: If $f(z)$ is analytic on and inside the closed contour C and z_0 is inside C, then

   $$\frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz - f(z_0) = 0.$$

   Proof: Let C_ρ be a circle of radius ρ centered at z_0, with ρ sufficiently small that C_ρ is entirely inside C.

   $$\frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz - f(z_0) = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)}{z - z_0}dz - f(z_0) \tag{1}$$

   $$\frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)}{z - z_0}dz - f(z_0) = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0}dz \tag{2}$$

   Also, $\forall \epsilon > 0, \exists \rho > 0$ such that $|z - z_0| \leq \rho \longrightarrow |f(z) - f(z_0)| < \epsilon$, so (3)

   $$\left|\frac{1}{2\pi i}\int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0}dz\right| \leq \frac{1}{2\pi}\int_{C_\rho} \left|\frac{f(z) - f(z_0)}{z - z_0}\right| |dz| \tag{4}$$

   $$\frac{1}{2\pi}\int_{C_\rho} \left|\frac{f(z) - f(z_0)}{z - z_0}\right| |dz| < \frac{1}{2\pi}\int_{C_\rho} \left|\frac{\epsilon}{\rho}\right| |dz| \tag{5}$$

   $$\frac{1}{2\pi}\int_{C_\rho} \left|\frac{\epsilon}{\rho}\right| |dz| \leq \epsilon \tag{6}$$

   Therefore $\frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz - f(z_0) = 0.$ (7)

3. (20 pts) Find the general form of the requested series **and** give its radius of convergence

   a) Find the Taylor series of $f(z) = \frac{z-1}{z+2i}$ centered at $z = 1$.

   b) Find the Laurent Series centered at $z = 0$ of $f(z) = \sin(\frac{1}{z})$

4. (15 pts) Evaluate $\int_{-\infty}^{\infty} \frac{1}{(x^2+16)^2}dx$

5. Evaluate $\int_0^\infty \frac{x^{1/4}}{x^2+9}dx$

6. Consider the function $G(z) = \frac{cos(\pi z)}{z^2 sin(\pi z)}$. You may use the identities:
   $\cos(z) = \cos x \cosh y - i \sin x \sinh y$ and
   $\sin(z) = \sin x \cosh y + i \cos x \sinh y$.

   a) Find and classify all the singularities of $G(z)$.

   b) Find the residues at all the singularities of $G(z)$ (you may use the table provided at the end of the exam).

   c) Consider the contour S made of a square centered at the origin, extending from $-(n+1/2)$ to $(n+1/2)$, with $n \in \mathbf{N}$, in both the x and y direction. Bound $\int_S G(z)dz$ as a function of n.

   d) Take the limit as $n \to \infty$ and use the Residue theorem, to obtain a famous identity.

7. The function
   $$f(z) = \sum_{n=2}^{\infty} \frac{n\, i^n\, z^n}{(n-1)!}$$
   is analytic and entire. Using integration or derivation of known Taylor Series, and multiplication/addition/subtraction of polynomials, find a closed formed expression for $f(z)$.

8. Using a logarithmic mapping, find the steady state temperature (satisfying $\nabla^2 T = 0$) in the region between the lines $y = x$ and $y = \sqrt{3}\, x$, in the first quadrant, if the temperature is $T = 2$ along $y = x$ and $T = 3$ along $y = \sqrt{3}\, x$.

9. Find the mapping required to transform the quarter-circle centered at $1-2i$ going from $3-2i$ to 1, into the lower half-circle of radius 1 centered at the origin.

$$\begin{aligned}
\frac{d}{dz}\left(\frac{\cos(\pi z)}{z\sin(\pi z)}\right) &= -(\cot(\pi z)/z^2) - (\pi\csc^2(\pi z))/z \\
\frac{d^2}{dz^2}\left(\frac{\cos(\pi z)}{z\sin(\pi z)}\right) &= 2\cot(\pi z))/z^3 + (2\pi\csc^2(\pi z))/z^2 + (2\pi^2\cot(\pi z)\csc^2(\pi z))/z \\
\frac{d^3}{dz^3}\left(\frac{\cos(\pi z)}{z\sin(\pi z)}\right) &= (-6\cot(\pi z))/z^4 - (6\pi\csc^2(\pi z))/z^3 - (6\pi^2\cot(\pi z)\csc^2(\pi z))/z^2 - \\
&\quad (4\pi^3\cot(\pi z)^2\csc^2(\pi z))/z - (2\pi^3\csc^4(\pi z))/z \\
\frac{d}{dz}\left(\frac{\cos(\pi z)}{\sin(\pi z)}\right) &= -(\pi\csc^2(\pi z)) \\
\frac{d^2}{dz^2}\left(\frac{\cos(\pi z)}{\sin(\pi z)}\right) &= 2\pi^2\cot(\pi z)\csc^2(\pi z) \\
\frac{d^3}{dz^3}\left(\frac{\cos(\pi z)}{\sin(\pi z)}\right) &= -4\pi^3\cot^2(\pi z)\csc^2(\pi z) - 2\pi^3\csc^4(\pi z) \\
\frac{d}{dz}\left(\frac{z\cos(\pi z)}{\sin(\pi z)}\right) &= \cot(\pi z) - \pi z\csc^2(\pi z) \\
\frac{d^2}{dz^2}\left(\frac{z\cos(\pi z)}{\sin(\pi z)}\right) &= -2\pi\csc^2(\pi z) + 2\pi^2 z\cot(\pi z)\csc^2(\pi z) \\
\frac{d^3}{dz^3}\left(\frac{z\cos(\pi z)}{\sin(\pi z)}\right) &= 6\pi^2\cot(\pi z)\csc^2(\pi z) - 4\pi^3 z\cot^2(\pi z)\csc^2(\pi z) - 2\pi^3 z\csc^4(\pi z) \\
\frac{d}{dz}\left(\frac{z^2\cos(\pi z)}{\sin(\pi z)}\right) &= 2z\cot(\pi z) - \pi z^2\csc^2(\pi z) \\
\frac{d^2}{dz^2}\left(\frac{z^2\cos(\pi z)}{\sin(\pi z)}\right) &= 2\cot(\pi z) - 4\pi z\csc^2(\pi z) + 2\pi^2 z^2\cot(\pi z)\csc^2(\pi z) \\
\frac{d^3}{dz^3}\left(\frac{z^2\cos(\pi z)}{\sin(\pi z)}\right) &= -6\pi\csc^2(\pi z) + 12\pi^2 z\cot(\pi z)\csc^2(\pi z) - 4\pi^3 z^2\cot^2(\pi z)\csc(\pi z) - 2\pi^3 z^2\csc^4(\pi z)
\end{aligned}$$