# Applied Math Preliminary Exam: Linear Algebra

University of California, Merced, January 2013

**Instructions:** This examination lasts 4 hours. Show explicitly steps and computations in your solutions. Credit will not be given to answers without explanation. Partial credit will be awarded to relevant work.

**Problem 1.** Given the system of linear equations

$$\begin{array}{ccccccccc} & & 6x_2 & + & 2x_3 & + & 10x_4 & = & b_1 \\ x_1 & + & x_2 & + & 4x_3 & - & 2x_4 & = & b_2 \\ x_1 & - & 2x_2 & + & 3x_3 & - & 7x_4 & = & b_3 \end{array}$$

(a) Find all possible values of $b_1, b_2$, and $b_3$ for which this system has solutions;

(b) Find all possible solutions of this system if $b_1 = 6$, $b_2 = 7$, and $b_3 = 4$.

**Problem 2.** Let $a, b$, and $c \in \mathbb{R}$.

(a) For what values of $a$ is the following matrix positive definite?

$$\begin{bmatrix} a & 0 & -2 \\ 0 & a & 2 \\ -2 & 2 & a \end{bmatrix}$$

(b) For what values of $a, b$, and $c$ does the following matrix have orthogonal rows?

$$\begin{bmatrix} a & -2 & 2 \\ 1 & b & 1 \\ -3 & 3 & c \end{bmatrix}$$

Verify your answer.

(c) The matrix in Part (b) has orthogonal rows but not orthogonal columns. Prove that any square matrix with ortho*normal* rows must also have orthonormal columns.

**Problem 3.** Find an orthogonal basis for the range space of $A$, where $A$ is given by

$$A = \begin{bmatrix} 5 & 6 & 1 \\ 2 & 4 & 0 \\ -1 & 2 & -1 \\ -1 & 2 & 1 \\ 1 & -2 & 3 \end{bmatrix}.$$

Verify that the basis vectors you found are orthogonal.

**Problem 4.** Let $A \in \mathbb{R}^{m \times n}$ with rank $k$.

(a) Let $A = U \Sigma V^T$ be the singular-value decomposition of $A$ with $U \in \mathbb{R}^{m \times m}, \Sigma \in \mathbb{R}^{m \times n}$, and $V \in \mathbb{R}^{n \times n}$. What do the rows and columns of $U$ and $V$ signify?

(b) How would you use the $QR$ factorization of $A$ to solve the least-squares problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad \|Ax - b\|_2,$$

where $b \in \mathbb{R}^m$?

**Problem 5.** Let $v \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

(a) What is the determinant of the matrix $M = I - \alpha v v^T$, where $I$ is the $n \times n$ identity matrix?

(b) For what values of $\alpha$ is $M$ nonsingular? Explain.

**Problem 6.** Let $A, B \in \mathbb{R}^{n \times n}$. Using the definition of a matrix norm, show that for any $1 \le p \le \infty$,

$$\|ABx\|_p \le \|A\|_p \|B\|_p \|x\|_p.$$

(**Hint:** Use the definition twice.) Conclude that $\|AB\|_p \le \|A\|_p \|B\|_p$ and that the condition number of any invertible matrix is at least 1.

**Problem 7.** Let $A \in \mathbb{R}^{m \times n}$ have full column rank, and let $b \in \mathbb{R}^m$ with $b \neq 0$.

(a) True or false: $m < n$. Explain.

(b) Prove from first principles that if $b \in \text{Range}(A)$, then $b \notin \text{Null}(A^T)$. Conclude that $A^TA$ is invertible by showing $(A^TA)x = 0$ if and only if $x = 0$.

(c) Show that $P = A(A^TA)^{-1}A^T$ is a projection matrix. Onto what space does $P$ project? Explain.

**Problem 8.** Let $A$ and $B$ be similar matrices. Which of the following are the same for both:

i. Eigenvalues

ii. Eigenvectors

iii. Trace

iv. Column space

v. Rank

Explain.

**Problem 9.** True or false (short explanation).

(a) Let $a, b, c \in \mathbb{R}^3$ be linearly independent. If $a \perp b$ and $b \perp c$, then $a \perp c$.

(b) Let $A \in \mathbb{R}^{n \times n}$. The sum of the eigenvalues of $A$ is real.

(c) Let $A \in \mathbb{R}^{m \times n}$. Null$(A)$ is a vector space.

(d) If $A$ is invertible, it must also be diagonalizable.