Duration: 4 Hours

Instructions: Show your work, credit will not be given to answers without explanation. Partial credit will be awarded for correct work, unless otherwise specified. When you are asked to explain yourself, please write clearly and use complete sentences. Good luck!

When you are asked to *explain* your reasoning, you must use complete sentences.

Notation: In the questions below, if A is a matrix, we use the following notation:

$C(A)$ = Column space of A.

$N(A)$ = Null space of A.

1. True or False: Provide a short explanation for each case. (Remember a true statement must ALWAYS be true.)

   (a) Every subspace of $\mathbb{R}^4$ is the nullspace of some matrix.

   (b) Let $\vec{b} \in \mathbb{R}^n$ and define $S = \{\vec{y} \in \mathbb{R}^n | \vec{y}^T\vec{b} = 1\}$. S is a subspace of $\mathbb{R}^n$.

   (c) If A is an $n \times n$ diagonalizable matrix, then 0 can not be an eigenvalue of A.

   (d) If A is an $n \times m$ matrix with orthonormal columns, then $AA^T = I$ where I is the identity matrix.

2. Consider the following linear system $A\vec{x} = \vec{b}$ where,

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 3 & a \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 2 \\ 3 \\ b \end{bmatrix}$$

   for constants a and b.

   (a) For what values of a and b will the system have infinitely many solutions?

   (b) For what values of a and b will the system have no solutions?

3. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$, compute $Exp(At)$.

4. Consider $\sin(x), \cos(x)$ and the inner product:

$$< f, g >= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx.$$

   (a) Show that $\sin(x)$ and $\cos(x)$ are orthonormal vectors with respect to this inner product.

   (b) Compute: $\| \sin(x) + \cos(x) \|$ where $\| \|$ is the norm induced by this inner product.

5. Consider the following matrix:

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

   (a) Find the singular value decomposition (SVD).

   (b) From the SVD indicate the corresponding orthonormal bases for each: $C(A), N(A), C(A^T)$ and $N(A^T)$.

6. Let A be an $n \times m$ matrix with rank r. Let P be the matrix which projects $\mathbf{R}^n$ onto $C(A)$ and S be the matrix which projects $\mathbf{R}^n$ onto $N(A^T)$. Show that $PS = 0_n$ (where 0_n is the $n \times n$ matrix of all 0's).

7. Let $\vec{x}, \vec{y}$ be nonzero vectors in $\mathbb{R}^n$, $n \geq 2$, and let

$$A = \vec{x}\vec{y}^T.$$

   (a) Find the determinant of A.

   (b) Find the trace of A.

   (c) Find the eigenvalues of A and their multiplicity.

   (d) Find the eigenvectors of A.

   (e) When is A diagonalizable?

8. Let A be an $m \times n$ matrix.

   (a) If Q Is an orthogonal matrix, prove

$$\|QA\|_2 = \|A\|_2.$$

   (b) Using what you proved in part (a). If A has a singular value decomposition $U\Sigma V^T$. Prove that:

$$\|A\|_2 = \|\Sigma\|_2 = \sigma_1,$$

   where σ_1 is the first (i.e., largest) singular value of A.

   Note that $\| \, \|_2$ is the usual 2-norm for matrixes. That is,

$$\|A\|_2 = \max_{\vec{x} \neq 0} \frac{\|A\vec{x}\|_2}{\|\vec{x}\|_2}$$

   where $\|x\|_2$ is the usual Euclidean norm of a vector in $\mathbb{R}^n$.

9. Let A be an $n \times n$ symmetric matrix.

   (a) Prove that the eigenvalues of A must be real.

   (b) If $\vec{x}^T A \vec{x} \geq 0$ for all $\vec{x}$ find a lower bound for the eigenvalues of A.