Applied Math Preliminary Exam: Linear Algebra

University of California, Merced, January 2012

Instructions: This examination lasts 4 hours.

- Show explicitly steps and computations in your solutions. Credit will not be given to answers without explanation.
- Partial credit will be awarded to relevant work.

Problem 1. Find dimensions and bases for the four fundamental subspaces: column space, null space, row space, and left null space for

$$A = \begin{bmatrix} -1 & 8 & -7 & -9 \\ -1 & -1 & -1 & 0 \\ 0 & -3 & 2 & 3 \end{bmatrix}.$$

Problem 2. Given the system of linear equations

$$\begin{array}{rcrcrcrcl} -5x_1 & - & x_2 & - & 2x_3 & + & x_4 & = & b_1 \\ 3x_1 & + & x_2 & + & x_3 & & & = & b_2 \\ -x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & b_3 \end{array}$$

(a) Find all possible values of b_1, b_2, and b_3 for which this system has solutions;

(b) Find all possible solutions of this system if $b_1 = 3$, $b_2 = -2$, and $b_3 = 0$.

Problem 3. Let

$$A = \begin{bmatrix} 4 & 0 \\ 2 & -5 \\ -6 & 1 \\ -2 & 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Find $\hat{x}$ that solves $\min_{x \in I\!R^4} \|Ax - b\|_2$.

(b) Describe how the singular value decomposition of A can be used to solve (a).

Problem 4. Prove the following:

(a) If λ is an eigenvalue of a Hermitian matrix A, then λ must be real.

(b) Let A be a *nilpotent* matrix, i.e., $A^p = 0$ for some integer $p \geq 1$. Then all of the eigenvalues of A are 0.

Problem 5. Let $v \in I\!R^n$ with $\|v\|_2 = 1$.

(a) Show that $P = vv^T$ is a projection matrix.

(b) Determine the eigenvalues and the corresponding eigenvectors of P.

Problem 6. Suppose v_1, v_2, and v_3 are vectors in $I\!R^4$. Describe each step (mathematically and in detail) how you would compute an orthonormal basis for the subspace spanned by v_1, v_2, and v_3.

Problem 7. Show that the largest eigenvalue $\lambda^\star$ of a symmetric matrix A solves

$$\lambda^\star = \text{maximize } x^T A x \quad \text{subject to } x^T x = 1.$$

Problem 8. State whether each of the following statements is true or false. Briefly explain why.

a. If $y \in I\!R^n$ and $A \in I\!R^{m \times n}$, then the decomposition $y = y_N + y_R$, where y_N is in the null space of A and y_R is in the range space of A^T, is unique.

b. If A and B are similar, then the trace of A is equal to the trace of B.

c. If A is positive definite, then A has only positive eigenvalues.

d. If $x, y \in \mathbb{R}^n$, then $\|x - y\|_2 \geq | \ \|x\|_2 - \|y\|_2 \ |$.