Applied Math Preliminary Exam: Linear Algebra
University of California, Merced, January 2012

Instructions: This examination lasts 4 hours.
- Show explicitly steps and computations in your solutions. Credit will not be given to answers without explanation.
- Partial credit will be awarded to relevant work.

Problem 1. Find dimensions and bases for the four fundamental subspaces: column space, null space, row space, and left null space for
\[ A = \begin{bmatrix} -1 & 8 & -7 & -9 \\ -1 & -1 & -1 & 0 \\ 0 & -3 & 2 & 3 \end{bmatrix}. \]

Problem 2. Given the system of linear equations
\[-5x_1 - x_2 - 2x_3 + x_4 = b_1 \\
3x_1 + x_2 + x_3 = b_2 \\
-x_1 + x_2 - x_3 + 2x_4 = b_3 \]
(a) Find all possible values of \( b_1, b_2, \) and \( b_3 \) for which this system has solutions;
(b) Find all possible solutions of this system if \( b_1 = 3, b_2 = -2, \) and \( b_3 = 0. \)

Problem 3. Let
\[ A = \begin{bmatrix} 4 & 0 \\ 2 & -5 \\ -6 & 1 \\ -2 & 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}. \]
(a) Find \( \hat{x} \) that solves \( \min_{x \in \mathbb{R}^4} \|Ax - b\|_2. \)
(b) Describe how the singular value decomposition of \( A \) can be used to solve (a).

Problem 4. Prove the following:
(a) If \( \lambda \) is an eigenvalue of a Hermitian matrix \( A \), then \( \lambda \) must be real.
(b) Let \( A \) be a nilpotent matrix, i.e., \( A^p = 0 \) for some integer \( p \geq 1 \). Then all of the eigenvalues of \( A \) are 0.

Problem 5. Let \( v \in \mathbb{R}^n \) with \( \|v\|_2 = 1 \).
(a) Show that \( P = vv^T \) is a projection matrix.
(b) Determine the eigenvalues and the corresponding eigenvectors of \( P \).

Problem 6. Suppose \( v_1, v_2, \) and \( v_3 \) are vectors in \( \mathbb{R}^4 \). Describe each step (mathematically and in detail) how you would compute an orthonormal basis for the subspace spanned by \( v_1, v_2, \) and \( v_3 \).

Problem 7. Show that the largest eigenvalue \( \lambda^* \) of a symmetric matrix \( A \) solves
\[ \lambda^* = \maximize x^T Ax \quad \text{subject to} \quad x^T x = 1. \]

Problem 8. State whether each of the following statements is true or false. Briefly explain why.
(a) If \( y \in \mathbb{R}^n \) and \( A \in \mathbb{R}^{m \times n} \), then the decomposition \( y = y_N + y_R \), where \( y_N \) is in the null space of \( A \) and \( y_R \) is in the range space of \( A^T \), is unique.
(b) If \( A \) and \( B \) are similar, then the trace of \( A \) is equal to the trace of \( B \).
c. If $A$ is positive definite, then $A$ has only positive eigenvalues.

d. If $x, y \in \mathbb{R}^n$, then $\|x - y\|_2 \geq |\|x\|_2 - \|y\|_2|$. 