Duration: 240 minutes

Answer all questions. Partial credit will be awarded to correct, but partial work. Points will be deducted for non-sensical answers. This test is meant to be difficult, so you are not expected to be able to answer every question perfectly.

1. Consider the initial-value problem \( y' + P(x)y = x, \ y(0) = 1 \), where

\[
P(x) = \begin{cases} 
1 & 0 \leq x \leq 2, \\
3 & x > 2. 
\end{cases}
\]

(a) Comment on the existence and uniqueness of solutions for this initial-value problem.

(b) Find a reasonable solution to this problem, i.e. one that is continuous for \( x \in [0, \infty) \).

(c) Sketch the graph of this solution.

2. Solve \( y' = \exp(x^2)/y^2 \) with initial condition \( y(0) = 1 \).

3. Solve \( y'' - y' - 2y = \cos x - \sin 2x \) with initial conditions \( y(0) = -7/20 \), and \( y'(0) = 1/5 \).

4. Solve \( y''' + y' = 0 \) with initial conditions \( y(0) = 0 \), \( y'(0) = 1 \), and \( y''(0) = 2 \).

5. A generalized Ricatti equation takes the form \( y' = P(x)y^2 + Q(x)y + R(x) \).

(a) Suppose \( y = u(x) \) is a solution of this equation. Show that \( y = u + 1/v \) reduces the generalized Ricatti equation to a linear equation in \( v \).

(b) Given that \( u(x) = x \) is a solution of \( y' = x^3(y - x)^2 + y/x \), use your result from part (a) to find all other solutions to this equation.

6. Consider the van der Pol oscillator governed by the equation

\[
\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0.
\]

(a) Determine how the stability of the zero solution depends on the non-negative parameter, \( \epsilon \).

(b) If a pendulum is governed by this equation, apply your stability result from (b) to describe the behavior of this pendulum.

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7. The equation \( P(x)y'' + Q(x)y' + R(x)y = 0 \) is said to be **exact** if it can be written in the form \( [P(x)y']' + [f(x)y] = 0 \), where \( f(x) \) is to be determined in terms of \( P(x), Q(x), \) and \( R(x) \). Find the necessary condition for exactness and then give the method of solution.

8. Locate and classify the singular points of the following differential equations.

   (a) \((x - 1)y'' + \sqrt{x}y = 0, \ x \geq 0\).

   (b) \(y'' + y'\log x + xy = 0, \ x \geq 0\).

   (c) \(xy'' + y\sin x = 0\).

   (d) \((x^2 - x)y'' + xy' + 7y = 0\).

9. Find two linearly independent solutions of \( xy'' + (1 + x)y' + 2y = 0 \), valid near \(x = 0\). It is sufficient to obtain the first three nonvanishing terms in the infinite series.

10. Find all eigenvalues and eigenfunctions for the Sturm-Liouville problem

    \[ x^2y'' + xy' + \lambda y = 0, \quad y(1) = y(b) = 0, \quad b > 1. \]