Applied Math Preliminary Exam: Complex Analysis
University of California, Merced, January 2016

Instructions: This examination lasts 4 hours. Show explicitly steps and computations in your solutions. Credit will not be given to answers without explanation. Partial credit will be awarded to relevant work. Assume $z = x + iy$, where $x, y \in \mathbb{R}$ wherever relevant.

Problem 1. (5 points each)
(a) Define precisely what it means for a function $f : \mathbb{C} \to \mathbb{C}$ to be analytic at $z_0$.
(b) Write $\frac{4 + 4\sqrt{3}i}{1 - \sqrt{3}i}$ in polar notation, $re^{i\theta}$, where $r \geq 0$ and $-\pi < \theta \leq \pi$.
(c) Show that for $z \in \mathbb{C}$, $|z|^2 = |z|^2$.
(d) Find all the values of $(4i)^{1/4}$.

Problem 2. (5 points each)
(a) Give an example of a complex function that is differentiable everywhere on the complex plane and is bounded. Explain.
(b) Show that $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ for $z_1, z_2 \in \mathbb{C}$.
(c) Let $n$ be an integer. Compute $\int_0^{2\pi} e^{i n \theta} d\theta$.

Problem 3. (20 points total)
(a) (5 points) Find and identify the kind of singularities (pole, removable, or essential) of the function
$$f(z) = \frac{(1 - \cos z)e^{1/(z-2)}}{z^2(z + 2)^2}$$
(b) (15 points) Find the Laurent series for
$$f(z) = \frac{1}{(z + 4)(z + 2)}$$
valid for $2 < |z| < 4$.

Problem 4. (15 points) Evaluate the integral
$$\int_{-\infty}^{\infty} \frac{x \sin x}{1 + x^2} dx.$$

Problem 5. (5 points each)
(a) Find a fractional linear transformation that maps 0 to $i$, 1 to 2 and $-1$ to 4.
(b) Describe mathematically and draw what happens to the line $y = x + 1$ under the mapping $f(z) = 1/z$.
(c) Describe mathematically and draw what happens to the half-disc $D = \{z \in \mathbb{C} : \text{Im}(z) \geq 0 \text{ and } |z| \leq 1\}$ under the mapping
$$f(z) = \frac{-iz + i}{z + 1}.$$
Problem 6. (5 points each) Short answers.

(a) Let $C$ be a circle centered at $z = 0$ with radius 4. Compute $\int_C \frac{e^z}{z-1} \, dz$. Explain.

(b) Define precisely the residue of a function $f(z)$ at an isolated singular point $z_0$, and explain its significance.

(c) Find all the points where $f(z) = y + ix$ is differentiable. Also find where it is analytic.