<u>Instructions:</u> Show all your work. Credit will awarded largely for the **quality of your work** and the **clarity of your explanations**.

1. (18 points) Answer each of the following questions.

   (a) Find the general solution of the ODE

   $$y' - \tan(x)y = \sin(x) \ .$$

   (b) Solve the IVP

   $$y' = -\sqrt{1-y^2} \ , \quad y(0) = 1 \ .$$

   (c) Let $g(t) \in C^1$ and let $y(t)$ satisfy the integral equation

   $$y(t) = e^{g(t)} + \int_0^t g'(s)y(s)ds.$$

   Convert this equation into an IVP for $y(t)$ and solve it.

2. (15 points) According to Aphrodite's Law, World Love grows as

   $$\frac{dL}{dt} = \heartsuit(A - L) \ ,$$

   where t is in Heavenly Days, the constant A is the Ambient Love, and $\heartsuit$ is the Holy Coefficient. On the first day, World Love is 0. By the sixth day, Adam and Eve have been created; and World Love is 1. On the eleventh day, Adam and Eve discover that World Love is $1 + \epsilon$, where $\epsilon \in (0, 1)$.

   (a) What is the Holy Coefficient $\heartsuit$?

   (b) What does World Love approach at $t \to \infty$?

   (c) What can you conclude from this story if $\epsilon = 1$?

3. (15 points) Solve the IVP

   $$y'' + \omega^2 y = \cos(\omega t) \ , \quad y(0) = 0, \ y'(0) = 0 \ ,$$

   where $\omega_0 > 0$.

4. (15 points) Find the general solution of system of ODEs

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + \mathbf{b}\,, \quad A = \begin{bmatrix} 3 & 3 \\ -8 & 13 \end{bmatrix}\,, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\,. \tag{1}$$

5. (15 points) Consider the nonlinear ODE

$$y'' + (1 - y^2)y' + y = 0\,.$$

(a) Find the equilibrium solutions (fixed points).

(b) Linearize the ODE around an equilibrium solution and classify its type of linear (in)stability, *i.e.*, its phase portrait in the (y, y') plane.

(c) Let $E(t) = y^2(t) + y'^2(t)$. Show that if $E(0) < 1$ then $E(t) < 1$ for all $t > 0$.

6. (15 points) Consider the ODE

$$x(1-x)y'' - 2(1-2x)y' - 6y = 0.$$

(a) Classify the type of singularity at $x = 0$.

(b) Use Frobenius' method to determine the indicial equation and the recurrence relation for a power series solution around $x = 0$.

(c) Obtain a polynomial solution.

7. (15 points) Consider the eigenvalue problem

$$y'' + \lambda y = 0\,, \quad y'(0) = 0,\ y'(\pi) = 0\,.$$

(a) Find the eigenvalues and corresponding eigenfunctions.

(b) If $\lambda = \lambda_n$ is one of the eigenvalues, what is the condition on a function $f(x)$ for the IVP

$$y'' + \lambda_n y = f(x)\,, \quad y'(0) = 0,\ y'(\pi) = 0$$

to have a bounded solution?

GOOD LUCK!