Preliminary Exam in Calculus – January 12, 2007

You have four hours to complete this exam. Each problem is worth 10 points.

1. Calculate each limit if it exists or explain why it does not exist.

$$a)\ \lim_{x\to 0}\frac{\sin(\sin x)}{x} \qquad b)\ \lim_{x\to 0}\frac{e^{-\frac{1}{x^2}}}{x^5} \qquad c)\ \lim_{x\to\infty}\ln|\sin x| \qquad d)\ \lim_{x\to 1+}(\cos x)^{\frac{1}{x-1}}$$

2. Graph the functions: $a) y(x) = (x-1)(x-2)^2(x-3)^3$; $\quad b) y(x) = \tanh(e^{-x})$.

3. Find the derivative of $y(x) = \int_1^x f(x-t)\,dt$, where f is a C^1 function.

4. Let $y(x)$ be defined implicitly by $xy = \sin y$. Evaluate $\dfrac{dy}{dx}$ when $y = 0$.

5. Find the following indefinite integrals.

$$a)\ \int\frac{dx}{(x-2)^{\frac{1}{3}}} \qquad b)\ \int\frac{e^x\,dx}{\sqrt{1-e^{2x}}} \qquad c)\ \int e^{2x}\sin x\,dx \qquad d)\ \int\frac{4x\,dx}{1-4x^2}$$

6. Determine whether the following integrals exist. You do not need to find their values, but justify your answer appropriately.

$$a)\ \int_0^\infty \cos x\,dx \qquad b)\ \int_0^1\frac{dx}{\sin x} \qquad c)\ \int_0^1 \ln(\sin x)\,dx$$

7. Calculate the 2^{nd} order Taylor polynomials for the following functions.

   (a) $f(x) = \cos\sqrt{x}$ around $x_0 = 0$

   (b) $f(x,y) = x^6 - (x+y)^2 + y^6$ around $(x_0, y_0) = (0,0)$

   (c) $f(x,y) = \dfrac{1}{1-xy}$ around $(x_0, y_0) = (0,0)$

8. Find all the extremal points of the following functions and classify each of these points as maxima, minima, or saddle: $a)\, u(x,y) = x^2 + xy^2$; $\quad b)\, u(x,y) = x^2 - xy^2$.

9. Use Gauss's Divergence Theorem to calculate the surface integral $\displaystyle\int_{S_R}\vec{u}\cdot d\vec{S}$, where $\vec{u}(x,y,z) = (x-z)\vec{i} + (z-y)\vec{j} + (y-x)\vec{k}$ and S_R is the sphere of radius R around the origin.

10. Use Stokes' Theorem to calculate the contour integral $\displaystyle\oint_C \vec{u}\cdot d\vec{r}$, where $\vec{u}(x,y,z) = (xy^2z^2)\vec{i} + (x^2yz^2)\vec{j} + (x^2y^2z)\vec{k}$; (x,y,z) are measured from Earth's center, and the integration is carried over Earth's equator.