Name:________________________________

UC Merced Applied Math Graduate Preliminary Examination
Calculus Component
Spring 2009

You have four hours to complete this exam. There are ten problems total, each worth ten points. Show all work on separate sheets of paper, and circle your final answers, where appropriate. When finished, please staple your work behind these sheets, with your name written at the top. You are allowed one hand-written crib sheet, but no calculators or other study aides are allowed. Remember to explain your work clearly and legibly, so that you may receive maximum credit. Good luck!!

Problem 1

Compute each of the following indefinite integrals.

1. $\displaystyle\int x^2 e^{-3x}\,dx$ 2. $\displaystyle\int \frac{\sec^2\theta}{1-\tan\theta}\,d\theta$ 3. $\displaystyle\int \sqrt{1+\cos ax}\;dx$

Problem 2

Compute each of the following definite integrals or show that it is divergent. Be sure to explain your reasoning.

1. $\displaystyle\int_1^\infty \frac{\ln(\ln x)}{x}\,dx$ 2. $\displaystyle\int_{-1}^1 \frac{x+1}{(x^2)^{1/3}}\,dx$ 3. $\displaystyle\int_0^\infty e^{cx}|\cos bx|\,dx \quad b, c > 0.$

Problem 3

Compute the Taylor series of the function $f(x) = \ln x$ about the point $c = 2$. What is the upper limit of x for which this series is *absolutely* convergent? Justify your answer.

Problem 4

For the parametric curve $\mathbf{r}(s) = \frac{2\sqrt{2}}{3}s^{3/2}\,\hat{\mathbf{x}} + \frac{1}{2}s^2\,\hat{\mathbf{y}} + s\,\hat{\mathbf{z}}$, find the arclength from $s = 0$ to $s = 1$.

Problem 5

Use Stokes's Theorem to evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\hat{\mathbf{x}} + yz\,\hat{\mathbf{y}} + zx\,\hat{\mathbf{z}}$, and C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, oriented counterclockwise as viewed from above.

Problem 6

Evaluate the surface integral $\int_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz\ \hat{\mathbf{x}} - 2y\ \hat{\mathbf{y}} + 3x\ \hat{\mathbf{z}}$, and S is the sphere $x^2 + y^2 + z^2 = 4$ with outward orientation. You may work in any coordinate system (e.g. rectangular, spherical, etc.) you choose, and you may use the Divergence Theorem if you wish.

Problem 7

Consider the function

$$f(x) = |x|e^{-x^2/2}.$$

A. What kind of symmetry does this function display?

B. Compute both $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Be sure to explain your logic.

C. (i) Find the position x for all local extrema. Identify whether each extremum is a local maximum or minimum (justifying your answer.) (ii) Identify all intervals over which f is increasing and all intervals over which f is decreasing.

D. Identify any positions x at which the function is not differentiable.

E. Identify the intervals over which the function is concave up and the intervals over which the function is concave down.

F. Neatly sketch the function $f(x)$, illustrating all of the above properties.

Problem 8

Gravel is being dumped from a conveyor belt at a rate of 10 m^3/min, such that it forms a pile in the shape of a cone, where the diameter of the cone's base is equal to its height. How fast is the height of the pile increasing when the pile is 5 m high? (Hint: recall that the volume of a cone equals 1/3 the volume of the corresponding cylinder.)

Problem 9

Find the maximum value of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $x^4 + y^4 + z^4 = 1$.

Problem 10

Suppose you are an "ice cream" engineer. You are assigned the task of designing an ice cream cone that holds a volume V_c of ice cream, as measured when the level of the ice cream is flush with the top of the cone (i.e. the ice cream is not mounded above the top of the cone.) What is the height h and radius r of the cone that minimizes the total material used to make the cone? Assume the side of the cone is a uniform thickness, and remember, of course, that an ice cream cone has no top. Express your final result in terms of V_c. Finally, compute the ratio h/r. Does it depend on V_c?

Some potentially useful facts about a cone with height h and radius r:
Volume = (Volume of corresponding cylinder)/3,
Lateral surface area (i.e. surface area of the cone's side) = $\pi r\sqrt{r^2 + h^2}$,
Area of top circle = πr^2.