2010 Preliminary Examination in Calculus
UC Merced, Graduate Program in Applied Mathematics

You have four hours to complete this exam. There are ten problems total, each worth ten points. Show all work on separate sheets of paper, and circle your final answers, where appropriate. When finished, please staple your work behind these sheets, with your name written at the top. You are allowed one hand-written crib sheet, but no calculators or other study aides are allowed. Remember to explain your work clearly and legibly, so that you may receive maximum credit. Good luck!

Problem 1. For each of the following limits, either calculate the correct value or explain why the limit does not exist:

1. $\lim_{x \to 10} \frac{|x-10|}{x-10}$
2. $\lim_{x \to 1} \frac{1-\sqrt{x}}{1-x}$
3. $\lim_{x \to \infty} x \sin \frac{1}{x}$

Problem 2. Compute each of the following integrals:

1. $\int_0^{\pi} x \sin x \, dx$
2. $\int_0^1 e^{\sqrt{x}} \, dx$
3. $\int_0^{\pi/2} \frac{1-\sin x}{\cos x} \, dx$

Problem 3. Consider the function $f(x) = \dfrac{x-1}{x^2-5x+6}$.

1. Identify any positions x at which $f(x)$ is discontinuous. For each discontinuity, find the limit of $f(x)$ as x approaches the discontinuity from above and from below.
2. Find and classify all critical points of $f(x)$.
3. Identify the intervals over which the function is concave up and the intervals over which the function is concave down.
4. Sketch the graph of $f(x)$ from $x = -1$ to $x = 5$.

Problem 4. Consider the function $f(t) = e^t$.

1. Approximate $f(t)$ by a Taylor polynomial of degree 1 about $t = 0$.
2. Show that for $0 < t \leq 1$, the approximation you obtained always *underestimates* the true value of $f(t)$.

Problem 5. Let S be the unit sphere defined by the set of points (x, y, z) such that $x^2 + y^2 + z^2 = 1$. Consider the integral $\iint_S (2x + 2y + z^2) \, dS$.

1. First use the Divergence Theorem to evaluate the integral.
2. Now evaluate the integral directly.

Problem 6. Let γ be the triangle traced counterclockwise with the three vertices $(0,0)$, $(1,0)$, and $(1,1)$. Consider the line integral $\int_\gamma (x-y)\,dx + (x+y)\,dy$.

1. First use Green's Theorem to evaluate the integral.
2. Now evaluate the integral directly.

Problem 7. Find the maximum and minimum values of $f(x,y,z) = 2x + y + 4z$ subject to the constraint $x^2 + y + z^2 = 16$.

Problem 8. Let

$$f(x) = \int_x^\infty \frac{1}{y^2 + 2x}\,dy.$$

Find $f'(x)$. (Note: evaluating the integral in the definition of $f(x)$ is not necessary.)

Problem 9. Let

$$\mathbf{F}(x,y,z) = yz\,\hat{\mathbf{x}} + 2xz\,\hat{\mathbf{y}} + e^{xy}\,\hat{\mathbf{z}}.$$

Let C be the circle $x^2 + y^2 = 16$ located at $z = 5$ in three-dimensional space. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ using any method of your choice.

Problem 10. Explain whether each of the following series converges or diverges. If a given series does converge, find the value to which it converges.

1. $\displaystyle\sum_{n=1}^\infty \frac{2}{n^2 + 4n + 3}$
2. $\displaystyle\sum_{n=1}^\infty \log\left(1 + \frac{1}{n}\right)$
3. $\displaystyle\sum_{n=1}^\infty n^2 e^{-n}$