Applied Math Preliminary Exam: Calculus

UC Merced, January 2014

Directions: This examination lasts 4 hours.

Problem 1) (20 points) For each of the following problems evaluate the limit:

(a)

$$\lim_{\theta \to 0} \frac{\cot(\pi\theta)\sin\theta}{2\sec\theta,}$$

(b)

$$\lim_{x \to 0^+} (1 + \sqrt{(x)})^{1/\sqrt{x}},$$

(c)

$$\lim_{x \to 1^-} \frac{\sin x}{x(1-x)},$$

(d)

$$\lim_{(x,y) \to (0,0)} \frac{x^{7/3}}{x^2 + y^2}.$$

Problem 2) (15 points)

(a) Given $y(x) = xz \ln(x + y + z)$, compute $\frac{dy}{dx}$.

(b) Given $4x^2y - 3y = x^3 - 1$ compute $\frac{d^2y}{dx^2}$.

(c) Given $y(x) = \int_{\sin x}^{\cos x} xtdt$ compute $\frac{dy}{dx}$.

Problem 3) (20 points) Compute the following integral or show that it diverges:

(a)

$$\int \frac{6e^{1/x}}{x^2} dx$$

(b)

$$\int x \sinh(x) dx$$

(c)

$$\int_2^4 \frac{\sqrt{x^2 - 4}}{x} dx$$

(d)

$$\int_{-\infty}^{\infty} \frac{x}{\sqrt{x^2 + 9}}$$

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Problem 4) (10 points) A rectangular box is to be made from a piece of cardboard 24 inches long and 9 inches wide by cutting out identical squares from the four corners and turning up the sides (see Figure 1). Find the dimensions of the box of maximum volume. What is this volume?

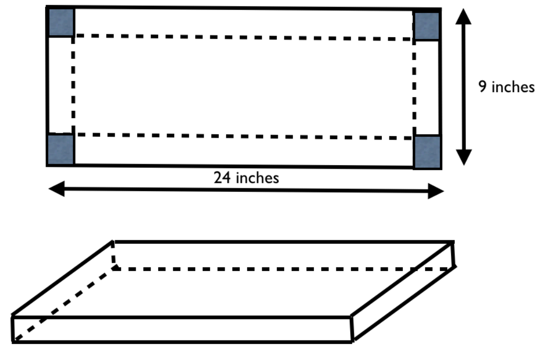

Figure 1

Problem 5) (10 points) Suppose $h'(x) = x^2(x-1)^2(x-2)$ and $h(0) = 0$. Sketch a graph of $y = h(x)$. (*Note:* you don't need to compute y exactly, just graph it!)

Problem 6) (10 points) Parametric equations for an object moving in the plane are $x = 3\cos t$ and $y = 2\sin t$, where t represents time and $0 \le t \le 2\pi$. Let P denote the object's position.

(a) Graph the path of P in xy plane.

(b) Find expressions for the velocity vector $v(t)$, speed $\|v(t)\|$ and acceleration vector $a(t)$.

(c) Find the maximum and minimum values of the speed and where they occur.

Problem 7) (10 points) Sketch the region bounded by the graphs $x = y^2$, $x = 0$ and $y = 3$ and show typical horizontal slice. Find the volume of the solid generated by revolving this region about the y-axis.

Problem 8) (10 points) If an object of rest mass m_0 has velocity v, then according to the theory of relativity its mass m is given by $m = m_0/\sqrt{1 - v^2/c^2}$, where c is the velocity of light. Explian how physicists get the approximation

$$m \approx m_0 + \frac{m_0}{2}\left(\frac{v}{c}\right)^2.$$

Problem 9) (10 points) Let $\mathbf{F}(\mathbf{r}) = \mathbf{F}(x, y, z) = (c^x \cos y + yz)\hat{\mathbf{x}} + (xz - e^x \sin y)\hat{\mathbf{y}} + xy\hat{\mathbf{z}}$ and C be a path defined by the half of a circle of radius 1 centered at $(0, 0, 0)$ for which $y = 0$ and $z \ge 0$ with direction from negative to positive x. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Problem 10) (10 points) The vector field $\mathbf{F}(x, y) = -\frac{1}{2}y\hat{\mathbf{x}} + \frac{1}{2}\hat{\mathbf{y}}$ is the velocity field of a steady counterclockwise rotation of a wheel about the z-axis. Calculate $\oint_C \mathbf{F} \cdot \mathbf{n}\,ds$ and $\oint_C \mathbf{F} \cdot \mathbf{T}\,ds$ for any closed curve C in the xy-plane where $\mathbf{n}$ is a unit normal vector pointing out of the region bounded by C and $\mathbf{T}$ is a unit tangent vector to C (*Note:* your answer can be nonnumeric but written in terms of characteristics of C).