Calculus Preliminary Exam *January 2015*

Duration: 240 minutes

Instructions: Answer all questions, without the use of notes, books or calculators. Partial credit will be awarded for correct work, unless otherwise specified. The total number of points is 100.

1. Determine as accurately as possible the limit as $n \to \infty$ of sequences with following general terms:

   (a) $a_n = \frac{e^n}{n!}$

   (b) $b_{n+1} = b_n^2 + 1/8$, with $b_1 = 1/2$

   (c)
   $$c_n = \sum_{k=0}^{k=n} \frac{1}{(\sqrt{k}+1)^2}$$

   (d)
   $$d_n = \sum_{k=0}^{k=n} (-1)^k \frac{x^{2k+1}}{(2k)!}$$

2. Compute the following limits:

   (a)
   $$\lim_{x \to 0} \frac{\sin x}{|x|}$$

   (b)
   $$\lim_{x \to 0} \left(\frac{\cos x}{x^2+4} \right) \left(\frac{e^x - \sin x - 1}{x^2} \right)$$

   (c)
   $$\lim_{x \to 0} \frac{(\sin(x+3))^2 - (\sin 3)^2}{x}$$

   (d)
   $$\lim_{(x,y) \to (0,0)} \frac{x^2 - y^4}{x^4 + y^4}$$

3. Calculate the following derivatives

   (a) $\frac{dy}{dx}$ given that $y^3 + 3y^2x + 3yx^2 + x^3 = x$.

   (b)
   $$\frac{d}{dt} \left(\int_0^{t^2} e^{tx^{1/2}} dx \right)$$

   (c) The price of an individual plane ticket, $p(t)$, the number of passengers on a plane, $n(t)$, and the expenses of flying the plane, $e(t)$, are all functions of time, t. Give an expression for the rate of change of the profits of operating this plane.

   (d) The coordinates of a point on a surface are given by $(x(r,s), y(r,s), z(r,s))$. Give a expression for the rate of change of the distance between such a point and the origin, with respect to changes in r.

4. Compute the following integrals

   (a) $\int e^{2x} \cos(3x) dx$

   (b) $\int \frac{dx}{\sqrt{9-x^2-6x}}$ for $-3-\sqrt{18} \le x \le -3+\sqrt{18}$.

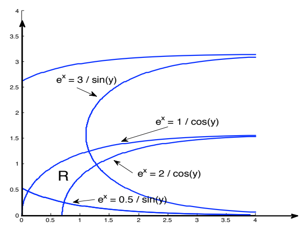
5. Surfaces and tangent planes

   (a) Sketch the surface give by $z = f(x, y) = 1 + x^2 + 2y^2 - 8y$.

   (b) Give an expression for the tangent plane going through a point $(x_0, y_0, f(x_0, y_0))$

6. Using polar coordinates, find the average value of $f(x, y) = y^3/(x^2 + y^2)^{3/2}$ over the triangle with vertices $(0, 0)$, $(2, 2)$ and $(-2, 2)$.

7. Integral in general coordinates Find the area of the region R enclosed by the curves $e^x = 0.5/\sin y$, $e^x = 3/\sin y$, $e^x = 1/\cos y$ and $e^x = 2/\cos y$, see figure below.



8. If the density of dust collected by a single passage of a vacuum cleaner is given by $d(x, y) = 5 + x^2 y \ (g/m)$, how much dust would be collected by a vacuum cleaner traveling along the upper half of a semi-circle of radius 2 centered at the origin?

9. Consider the part of the surface $z = 4 - x^2 - y^2$ (in meters) above the circle in the xy–plane of radius 1m centered at the origin, oriented with its normal pointing up. What is the flux of water into that surface if the water velocity is $\vec{F}(x, y, z) = (2x, 2y, -4z + 1)$m/s? How long would it take to get 31.416 m^3 of water through this surface?

10. General questions

    (a) Give two properties of the gradient of a function.

    (b) Give the formula of a function of one variable with a local minimum and a horizontal asymptote at $y = 2$.

    (c) Explain (briefly) how Green's theorem is a special case of Stokes's theorem.

    (d) Give a definition of the divergence of a vector field in terms of flux through a surface.

    (e) If $\vec{F}$ is a velocity field, how would you interpret the meaning of curl $\vec{F}$?

    (f) Can there be a non-constant, periodic function with at least one horizontal asymptote? Explain.

    (g) Is the following integral equal to a finite number? Explain. $\int_0^\infty \frac{e^{-2x}}{x^{1/3}+x} dx$

    (h) Sketch and give the formula for the volume element in spherical coordinates.