Calculus Premilinary Exam *January 2016*

Duration: 240 minutes
Instructions: Answer all questions. The total number of points is 150.

1. (20 pts) Compute the following limits

   (a)
   $$\lim_{x\to a} \frac{e^{x^2} - e^{a^2}}{a - x}$$

   (b) Let x_n be the position of a walker after n steps. Starting from $x_0 = 0$, each step is taken to the right, with a length given by half the distance between the walker and the point $x = 2$. What is $\lim_{n\to\infty} x_n$?

   (c)
   $$\lim_{x\to 0} \frac{x}{\sin(1/x)}$$

   (d)
   $$\lim_{x\to 1} \frac{x^4 + 2x^3 - x^2 + x + 3}{x^4 - x^2 + 1}$$

   (e)
   $$\lim_{(x,y)\to(0,0)} f(x,y) \quad \text{with} \quad f(x,y) = \begin{cases} \frac{x^2+y^4}{x^2+y^2}, & x^2 + y^2 \neq 0 \\ 1, & \text{otherwise} \end{cases}$$

2. (8 pts) Take the following derivatives:

   (a) $\frac{dy}{dx}$ for $x^4 + y^4 = 1$

   (b) $\frac{d}{dx}\left(\int_{\sin x}^{x} \frac{e^{xt^2}}{t}\,\mathrm{d}t\right)$

3. (9 pts) Integrate the following:

   (a) $\int_0^\infty \frac{xe^{-x^2}}{2}\,\mathrm{d}x$

   (b) $\int \frac{x^2}{\sqrt{4-x^2}}\,\mathrm{d}x$

   (c) $\int \frac{24x}{4x^2-12x+9}\,\mathrm{d}x$

4. (6 pts) Show that for any integer n, the following relation holds: $\int_0^\infty x^n e^{-x}\,\mathrm{d}x = n!$

5. (20 pts) Determine whether the following finite sums and infinite Series are convergent, and when possible determine the value they converge to.

   (a)
   $$\sum_{n=1}^{100} \left(\frac{-1}{3}\right)^n$$

   (b)
   $$\sum_{n=1}^{\infty} \left(\frac{1}{n^{n/2} + 1}\right)^{1/n}$$

(c)

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n+1}$$

(d)

$$\sum_{n=0}^{\infty} 2^n \frac{(x-1/2)^n}{n!}$$

6. (12 pts) Let x be the number of hours spent studying math in a day, and y be the number of hours spent studying anything in a day. Let the accumulated knowledge be given by $f(x,y) = \frac{x^3}{3} - \frac{5x^2}{2} + 4x + 100 + \frac{y-2}{2}$. Determine appropriate constraints on x and y and find how many hours per day one should study mathematics and other topics to maximize one's knowledge (according to this model).

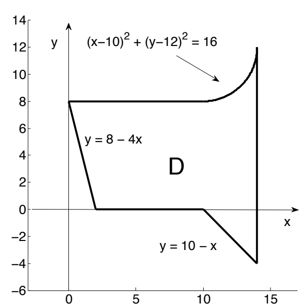
7. (8 pts) Find the distance between the point (1,3,-1) and the plane $x - 2y - z + 5 = 0$. If you use a specific distance formula, explain how it was obtained.

8. (10 pts) The integral $R = \int \int_D f(x,y)\ \mathrm{d}A$ calculates the total amount of rainfall in the United States in 2014.

   (a) Give the appropriate meaning of D, dA, and of the function $f(x,y)$.

   (b) If D is approximated as the region drawn below, set-up the proper bounds of integration in Cartesian coordinates.

   (c) Find the area of D.



9. (12 pts) The integral $M = \int \int \int_V h(x,y,z)\ dV$ calculates the mass of Uranus.

   (a) Assuming that the planet is a sphere centered at the origin, with radius 1, set up the integral for the mass in cylindrical coordinates.

   (b) In the same conditions, set up the integral for the mass in spherical coordinates.

   (c) In the same conditions, set up the integral for the mass in Cartesian coordinates.

10. (12 pts) Compute the flux of water into a vertical cylindrical fishing net if the velocity of the water is $\vec{F} = (xz\hat{\imath} + yz\hat{\jmath} + e^{\cos y}\hat{k})$ cm/min and the net lies along the z-axis below the plane z = 0 and has radius 4m and depth 5m. How much fish would you find in the net after one hour if there are 0.02 fish per meter cubed?

11. (12 pts) Consider the vector field $\vec{F} =< x^2 - y + 2z, y^2, 2x + 3y + z^2 >$, and the plane Π described by
$\vec{r}(u, v) =< u + v, 2u - 3v, u + 5v + 1 >$.

   (a) Describe in words the curve C given by $\vec{r}(u, v)$ subject to the constraint $\frac{(u-u_0)^2}{a^2} + \frac{(v-v_0)^2}{b^2} = 1$ for some constants a, b, u_0, and v_0.

   (b) Give the normal $\vec{n}$ to the plane Π.

   (c) Show that the work $W = \oint_C \vec{F} \cdot \frac{d\vec{r}}{dt} dt$ is independent of u_0 and v_0.

   (d) 5 pts BONUS: Determine the work W as a function of a and b.

12. (21 pts) Answer the following in no more than two lines of text or computations.

   (a) If you force more gas into a point than comes out of it, what operator applied to the gas velocity field $\vec{u}$ would capture the rate at which gas accumulates? What sign should the result be?

   (b) In Green's theorem, $\oint_C \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt = \int\int_A \mathrm{curl}\vec{F}\, dA$, give a **specific** example of what C and A may be.

   (c) If $\vec{c} = \vec{a} \times \vec{b}$, what can you say about the direction of $\vec{c}$ and the geometrical interpretation of $||\vec{c}||$?

   (d) Label each of the following objects as vectors, scalars, or meaningless:
   $\nabla f(x, y)$ $\quad$ $\vec{F}(x, y)\vec{v}$ $\quad$ $\nabla \times \vec{F}(x, y)$ $\quad$ $\vec{v} \cdot \vec{F}(x, y)$ $\quad$ $\nabla \cdot \nabla f(x, y)$

   (e) Give a constant approximation, and, separately, a linear approximation to $f(x, y) = e^x \cos y + 2y + 3$, at the point $(0, 0)$.

   (f) Consider a function $T(t, z)$ representing the temperature outside this classroom as a function of time, t, and elevation, z. Explain in words the meaning of $\frac{\partial T}{\partial z}$ and give its expected sign.

   (g) What surface is described by $\vec{r}(u, v) =< u, 2\cos v, 2\sin v >$?