Calculus Preliminary Exam *January 2017*

Duration: 240 minutes

Instructions: Answer all questions. The total number of points is 130.

1. (9 pts) Limits:
   (a) Give an example of a limit of a function that you think is best evaluated by plugging the limit point into this function. Evaluate this limit.
   (b) Give an example of a limit of a function that you think is best evaluated using L'Hospital's rule (where algebraic simplifications are not sufficient). Evaluate this limit.
   (c) Give an example of a limit of a function that you think is best evaluated using L'Hospital's rule for a different indeterminate form as in (b). Evaluate this limit.

2. (8 pts) Give an example of a use of the fundamental theorem of Calculus to calculate
   (a) a derivative
   (b) an integral.

3. (5 pts) Give a word problem such that an equation containing the derivative of a quantity may be obtained from the word description. Clearly indicate the notation you use.

4. (9 pts: 5,4) A line in the xy-plane is given by the equation $mx + b - y = 0$.

   (a) Derive a formula for the Euclidean distance between a point (x_0, y_0) and the line (denote it $d(m, b, x_0, y_0)$).
   (b) Give an equation describing the curve made of all the points that are located at an equal distance between the line $mx + b - y = 0$ and the point (x_0, y_0).

   For my curiosity (no points), do you know what is the curve obtained above?

5. (6 pts) Prove the formula $\frac{d(x^n)}{dx} = nx^{n-1}$, for $n \in \mathbb{N}$.

6. (9 pts) Give an example of an integral where the best technique to use is:
   (a) Substitution
   (b) Partial fractions
   (c) Integration by parts

7. (5 pts) Give an example of a Series which is convergent but not absolutely convergent.

8. (12 pts) Determine as accurately as you can the value of the following finite sums or infinite Series.

   (a) $\displaystyle\sum_{n=4}^{30} \left(-\frac{4}{3}\right)^n$ (b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n!}{n^n}\right)$ (c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x+1)^{2n}}{(2n)!}$

9. (6 pts) Explain the connection between Taylor Series and second derivative tests when classifying local extrema.

10. (12 pts) Consider the "heart" shape H with boundaries given by $y = 2x - 4$, $y = -2x - 4$, $y = \sqrt{1 - (x-1)^2}$, and $y = \sqrt{1-(x+1)^2}$.
    - (a) Sketch H.
    - (b) Set up an integral to determine the mass of H if its density is given by $\rho = 1 + x^2 + 2y^2$ (do not integrate).
    - (c) Determine the area of H.

11. (9 pts) The integral $M = \int\int\int_V h(x,y,z)\, dV$ calculates the mass of a grapefruit.
    - (a) Explain in words the meaning of V, $h(x,y,z)$, and dV.
    - (b) Assuming that the fruit is a sphere centered at the origin, with radius 6, set up the integral for its mass in cylindrical coordinates.
    - (c) In the same conditions, set up the integral for the mass in spherical coordinates.

12. (11pts) Consider the surface given by $4x + 2y + 3z - 24 = 0$.
    - (a) Give a point on this surface.
    - (b) Give two vectors parallel to this surface (but not parallel to each other).
    - (c) Give a parametrization of this surface.
    - (d) Give an expression for the flux through this whole (infinite) surface of the vector field $\vec{F} =< \frac{1}{z^4+y^4+1}, \frac{1}{x^4+z^4+1}, \frac{1}{x^4+y^4+1} >$. DO NOT EVALUATE THE INTEGRAL.

13. (6 pts) Give a line integral (and its domain) that is easier to compute using Green's theorem than using a direct parametrization.

14. (21 pts) Answer the following in no more than two lines of text or computations.
    - (a) Give an equation satisfied by all the points (x, y, z) such that any vector connecting (x, y, z) to the point $(3, 5, 1)$ is perpendicular to the vector $-\vec{i} + 5\vec{j} - 2\vec{k}$.
    - (b) Give a situation where the dot product is useful.
    - (c) Give the best local constant approximation, and, separately, linear approximation to $f(x, y) = x^2 \sin y + 2y + 3$, at the point $(1, 0)$.
    - (d) Consider a function $T(t, z)$ representing the temperature outside this classroom as a function of time, t, and elevation, z. Explain in words the meaning of $\frac{\partial T}{\partial z}$ and give its expected sign.
    - (e) If $\vec{F} =< y, x^2 >$, what is the maximum of $||\vec{F}||$ over the curve $\vec{r}(t) =< t, t^2 >$, for $1 \le t \le 2$?
    - (f) Sketch a 2D vector field whose divergence at the origin is positive.
    - (g) Parametrize a cylinder of radius 2 centered on the line $\vec{r}(t) =< t, 2t, 3t >$.