Calculus Preliminary Exam *January 2018*

Duration: 240 minutes
Instructions: Answer all questions, with the help of a single page of hand-written notes. Partial credit will be awarded for correct work. The total number of points is 100.

1. (15 pts, 3 each) Evaluate the following limits:
   (a)
   $$\lim_{x\to 0} \frac{x}{|x|}$$
   (b)
   $$\lim_{x\to 0} \frac{(\sin x)^2}{\cos(x) - 1}$$
   (c)
   $$\lim_{n\to\infty} n^{1/n}$$
   (d)
   $$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k}$$
   (e)
   $$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{n}$$

2. (9 pts, 3 each) Evaluate the following derivatives:
   (a)
   $$\frac{df}{dx} \quad \text{if} \quad f(x) = \log(\log(x))$$
   (b)
   $$\frac{dy}{dx} \quad \text{if} \quad x^4 + y^4 = 1$$
   (c)
   $$\frac{df}{dx} \quad \text{if} \quad f(x) = \int_{x^{1/2}}^{1} e^{-x^2} dx$$

3. (6 pts) Give a function of one variable, $f(x)$, that is everywhere differentiable and for which $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} \frac{f(x)}{x} = 1$

4. (5 pts) Give a function whose first derivative is continuous, but whose second derivative is not.

5. (6 pts) A household uses 100 plies/day of paper, which is wrapped around a cylinder. Each ply is 10cm long and 0.1cm thick. Derive a differential equation for the rate of change of the radius R of the paper roll over time.

6. (13pts: 5,3,5)
   Consider the line $\vec{r}(t) =< t, 2 + 3t, 2 - t >$ and the plane Π given by $x - y + 2z - 5 = 0$.
   (a) Does the line $\vec{r}(t)$ intersect the plane Π? If so, find where.

(b) Give a vector perpendicular to $\vec{r}(t)$ and parallel to the plane Π.

(c) Find the distance between the point $P =< 2, 3, 4 >$ and the plane Π.

7. (6 pts) A coordinate system is chosen so that Merced is at the origin, the y-axis points North and the x-axis points East. Explain in words the meaning of the directional derivative $Df_{\vec{v}}(2, 3) = 1$, with the vector $\vec{v} =< -1, 1 >$. Here distances are measured in miles, and the function $f(x, y)$ measures the number of cows per square mile, in thousands.

8. (11 pts 5,3,3) The integral $M = \int\int\int_V h(x, y, z)\, dV$ calculates the mass of a snowball

   (a) Explain in words the meaning of V, $h(x, y, z)$, and dV.

   (b) Assuming that the snowball is a sphere centered at the origin with radius 3, set up the integral for the mass in cylindrical coordinates.

   (c) In the same conditions, set up the integral for the mass in spherical coordinates.

9. (9 pts 3,3,3) Consider the surface $z = f(x, y) = e^{x^2 - y^2}$.

   (a) Sketch at least 3 contours of this function.

   (b) Sketch the surface $z = f(x, y)$.

   (c) Give one non-trivial fact (of your choice) about this function.

10. (8 pts) Sketch the region of integration and evaluate the iterated integral given below by reversing the order of integration

$$\int_0^4 \int_{x^{1/2}}^2 e^{y^3}\, dy\, dx$$

11. (11 pts 4,4,3) Find the work done by the force field $\vec{F} =< x - y, -x - 2y >$ on a particle traveling from $(-1, 2)$ to $(2, 3)$ along a straight line.

    (a) Using a direct parametrization.

    (b) Using the Fundamental theorem of line integrals (and finding the potential of $\vec{F}$).

    (c) Give a vector field for which one could not find the work using the Fundamental theorem of line integrals.

12. (13 pts 4,2,2,2,3) Consider the cube of side 2 aligned with the x, y, and z axis, and with a corner at the origin, as well as the velocity field of the wind $\vec{W} =< 1, 2, 3z >$. We are interested in the flux F of the wind coming out of the cube.

    (a) What do S, dS, and $\vec{W} \cdot \hat{n}$ stand for in the expression of the flux $F = \int\int_S \vec{W} \cdot \hat{n}\, dS$?

    (b) Explain why the flux adds up to 0 over all the vertical sides.

    (c) Give a parametrization of the top and bottom surfaces, including bounds on the parameters.

    (d) Compute the flux going coming out of the cube through the top and bottom surfaces.

    (e) Use the divergence theorem to compute the flux coming out of the cube