Directions: Duration: 4 hours. One crib sheet is allowed. Credit will not be given to answers without explanation. Partial credit will be awarded to relevant work. Enumerate the papers and do **not** staple them.

1. Sketch the graph of the function $y(x) = \ln\sqrt{\dfrac{x+1}{x-1}}$ for real values of x (where the function is defined).

2. Find each of the following limits, if it exists.

$$(a)\ \sum_{k=0}^{\infty} \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-k/n}, \qquad (b)\ \lim_{x\to 1+} e^{\frac{1}{x-1}}\ln(x)\,, \qquad (c)\ \lim_{(x,y)\to(0,0)} y\sin\frac{1}{x}\,.$$

3. For each of the functions defined below, seek an expression for $\dfrac{dy}{dx}$ as an explicit function of x alone or as an explicit function of both x and y.

   (a) $y(x) = \displaystyle\int_1^{\sqrt{x}} \operatorname{sech}(x)\,dx\,.$

   (b) $y(x)$ is defined implicitly by the equation $\ln x - \ln y = x - y\,.$

4. Let $f(x)$ be a continuous function from $[0,1]$ to itself. Prove that there is a point $x_\star \in [0,1]$, such that $f(x_\star) = x_\star$.

5. Give an example of a function, $f(x)$, that is differentiable for all $x > 0$, such that $\lim_{x\to\infty} f(x) = 0$ but $\lim_{x\to\infty} f'(x)$ does not exist.

6. Find each of the antiderivatives.

$$(a)\ \int \frac{x\,dx}{x^2-4}\,, \qquad (b)\ \int e^{-x}\cos 3x\,dx\,, \qquad (c)\ \int \frac{e^x}{\sqrt{1-e^x}}\,dx\,.$$

7. Determine whether the definite integral $\displaystyle\int_0^1 \frac{dx}{\ln(1+x)}$ exists. You do not need to find it.

8. Suppose that a sun is a spherical ball of radius $R_\odot$, whose mass density is radially symmetric and described by some function $\rho(r)$, where $r^2 = x^2+y^2+z^2$. Let $M(a)$ be the total mass of the portion of the sun that is a concentric ball of radius a, where $0 \le a \le R_\odot$. If you know that $M(a) = Ca^{5/2}$, where C is a constant, what is $\rho(r)$?

9. Find all the critical points of the function $u(x,y) = x^3 + y^2 - 6xy$ and classify them as maxima, minima or saddle.

10. What is the radius of convergence of the Taylor series of the function $f(x,y) = \dfrac{1}{1+\sin(x)\sin(y)}$ around the point $(x_0, y_0) = (0,0)$?

11. Let $\vec{\mathbf{u}}(x,y,z) = 2xe^{-z}\vec{\mathbf{i}} + 2ye^{-z}\vec{\mathbf{j}} - (x^2+y^2)e^{-z}\vec{\mathbf{k}}$. be the water velocity field around a cylindrical ocean coral of radius R, where z is measured from the ocean's bottom and $x = y = 0$ is the coral's center.

   (a) Show that $\vec{\mathbf{u}}$ is conservative and find its potential function, $\phi(x,y,z)$.

   (b) Provide an integral for the work done by the water on an angelfish swimming around the bottom of the coral.

12. Let $\vec{\mathbf{H}}(x,y,z) = (x-y)\vec{\mathbf{i}} + (y-z)\vec{\mathbf{j}} + (z-x)\vec{\mathbf{k}}$ be the magnetic field in and around a spherical ball of radius R. Calculate the flux $\displaystyle\oiint_S (\vec{\mathbf{H}}\cdot\vec{\mathbf{n}})ds$ on the ball's surface, where $\vec{\mathbf{n}}$ is the outward normal.