**Duration: 4 hours**
**Instructions:**

- Read the exam entirely BEFORE starting.
- Each exercise is independent, you can do them in a different order.
- Answer all questions, you may use one page, handwritten equation sheet.
- Partial credit will be awarded for correct work, unless otherwise specified.
- Please write **clearly** and use **complete sentences**.
- Credit will not be given to answers without explanation.
- Write each problem on a separate page. Please make sure you clearly mark the problem you are working on (e.g., 1a) ).
- No calculator or electronic devices allowed.
- Good luck!

1. Compute the following limits (if it doesn't exist, justify):

$$(a)\ \lim_{x\to 0}\frac{\tan x - x}{x^3},\qquad (b)\ \lim_{(x,y)\to(0,0)}\frac{xy^4}{x^2+y^8},\qquad (c)\ \lim_{n\to\infty}\left(\frac{n}{2}\right)^{2/n}$$

2. Show that

$$I_n := \int_0^1 (1-x^2)^n\,dx = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

*Hint: find a relation between $I_n$ and $I_{n-1}$.*

3. Find the antiderivative of the following integrals:

$$(a)\ \int \frac{x}{\sqrt{1-2x^2}}dx,\qquad (b)\ \int \frac{2x+1}{x^2+x-2}\,dx$$

4. For each series, explain if the series converges or not (bonus if you find the limit when it exists).

$$(a)\ \sum_{n=0}^{\infty}\frac{(x+2)^n}{(n+3)!},\qquad (b)\ \sum_{n=2}^{\infty}\ln\left(1-\frac{1}{n^2}\right),\qquad (c)\ \sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n+1}$$

5. Sketch the function $\frac{x^2}{x^2-4x+3}$. Provide as much information as possible.

6. Surfaces and tangent planes
   (a) Give an equation of the form $z = f(x,y)$ where $f(x,y)$ is not linear or constant and such that the tangent plane to that surface at $(1,-1)$ is $P(x,y) = 4+2x-y$. (READ PART B BEFORE ANSWERING).
   (b) Sketch the function you came up with in part (a).
   (c) Considering now $f(x,y) = 0$ with $f$ the function you came up with in part (a), find $\frac{dy}{dx}$.

7. Consider the function $f(x,y) = y^2+xy-2x^2$, and the region $D$ enclosed by $y-x=-1$, $y-x=2$, $y+2x=-1$, $y+2x=2$.
   (a) Sketch $D$.
   (b) Evaluate the volume over $D$ between $z = f(x,y)$ and $z=0$.

8. Compute $\iint_S \vec{F}\cdot d\vec{S}$ with the vector field $\vec{F} = \langle x^2, -2xy, 2z+3\rangle$, and $S$ the upper half-sphere of radius 1 centered at 0.

9. Consider the space curves $C_1$ given by $\vec{r}_1(t) = \langle t, t^2, t^3\rangle$, $0\le t\le 1$, $C_2$ given by $\vec{r}_2(t) = \langle (t^2-2t)/3, t^2-4t+4, t-2\rangle$, $2\le t\le 3$ and the vector field $\vec{F} = \langle \sqrt{x}+y, e^y+x, 3z^2\rangle$.
   (a) Show that $\vec{F}$ is conservative and find the potential.
   (b) Compute the work done by $\vec{F}$ along $C_1$, and compare it with the work done by $\vec{F}$ along $C_2$.

10. General questions
    (a) Explain why Green's theorem is a specific case of Stokes theorem.
    (b) If $\vec{c} = \vec{a}\times\vec{b}$, what can you say about the direction of $\vec{c}$ and the geometrical interpretation of $||\vec{c}||$?
    (c) Label each of the following objects as vectors, scalars, or meaningless:
    $\vec{\nabla}f(x,y)$ $\quad \vec{F}(x,y)\vec{v}$ $\quad \vec{\nabla}\times\vec{F}(x,y)$ $\quad \vec{v}\cdot\vec{F}(x,y)$ $\quad \vec{\nabla}\cdot\vec{\nabla}f(x,y)$
    (d) What surface is described by $\vec{r}(u,v) = < u, 2\cos v, 2\sin v >$?
    (e) For a given function $z = f(x,y)$, explain briefly the significance of the set of points satisfying $f(x,y) = k$, for $k\in\mathbb{R}$. Illustrate by sketching an example.