Duration: 4 hours (9am-1pm)

Instructions: Follow general instructions given on previous page

- Read the exam entirely BEFORE starting.
- Each exercise is independent, you can do them in a different order.
- Please write **clearly** and use **complete sentences**.
- No calculator or electronic devices (for computation purposes) allowed.
- Submit your work as documented previously.
- Good luck!

1. Evaluate the following limits (if it doesn't exist, justify):

$$(a)\ \lim_{x\to 1} \frac{\sqrt{x-1}}{1-x}, \qquad (b)\ \lim_{x\to 0^+} \frac{\sin(x)}{x(x+1)}, \qquad (c)\ \lim_{n\to\infty} \left(1+\frac{2}{n}\right)^{1/n}$$

2. Determine if the following series are convergent or divergent. In case of convergence, provide the limit.

$$(a)\ \sum_{n=0}^{\infty} \frac{(x-3)^n}{2^n n!}, \qquad (b)\ \sum_{n=2}^{\infty} \frac{4n^2+n}{10+2n^2}, \qquad (c)\ \sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

3. Integrate the following:

$$(a)\ \int \frac{x+1}{x^2-1} dx, \qquad (b)\ \int_0^{\frac{\pi}{2}} \sin^{16} x\, dx$$

(Hint: provide first a recurrence relation for b))

4. Take the following derivatives:

$$(a)\ \frac{d}{dx}\left(\int_0^{x^2} \sin(xt)\, dt\right) \qquad (b)\ \frac{dy}{dx} \quad \text{for} \quad y^4+2xy=1$$

5. Let C be a unit cube with vertices (0,0,0), (0,1,0), (1,0,0), (1,1,0), (0,0,1), (0,1,1), (1,0,1), (1,1,1), and consider the vector field $\vec{F} = \langle 2x+y, z+x, 0\rangle$. We denote S the surface of the cube, positively oriented.

   (a) Sketch C and S.

   (b) Use the Divergence Theorem to evaluate the integral $\iint_S \vec{F}\cdot d\vec{S}$ (we expect a clear verification of all hypotheses).

   (c) Now compute the integral $\iint_S \vec{F}\cdot d\vec{S}$ directly.

6. Consider the vector field $\vec{F} = \langle x^2+1, 3y, z\rangle$.

   (a) Show that $\vec{F}$ is conservative and find an associated potential function ϕ.

   (b) Use the previous question to compute the line integral $\int_C \vec{F}\cdot d\vec{r}$ where C is the segment going from the origin to the point $A=(2,1,1)$. (partial credit will be given if you decide to compute it directly if you haven't found the first question).

7. Consider an annulus $\mathcal{A}$ of inner radius 1, and outer radius 10, in the xy-plane. We want to compute $I = \int_{\mathcal{A}} f\, dA$, where f is a function given in polar coordinates $f(r,\theta) = \ln(r)\sin(\frac{\theta}{3})$.

   (a) Sketch the domain $\mathcal{A}$.

   (b) Compute I using Euler's coordinates $(r,\theta)\mapsto(z,\theta) = (\ln r, \theta)$.

8. Consider the plane $x+2y-z=3$

   (a) Provide a parameterization of that plane.

   (b) Given a point $A=(1,2,1)$, find a vector (starting at A) parallel to the plane, and a unit vector (starting at A) perpendicular to the plane.

(c) Find the distance from A to the plane.

9. Find a two-variables function f such that its tangent plane at $(1, 1)$ is given by $z = 3+(x-1)+2(y-1)$.

10. General questions

(a) What are the differences between Stokes' theorem and Green's theorem ? You may provide a sketch to illustrate.

(b) Provide a parameterization of the ellipsoid given by $4x^2 + 6y^2 + 2z^2 = 2$.

(c) Provide two vectors $\vec{a}$, $\vec{b}$ so that $\vec{a} \cdot \vec{b} = ||\vec{a} \times \vec{b}||$.

(d) Sketch some contour plots of the function $f(x, y) = x^2 + y^2 - 2xy$.

(e) Explain in a few words what is a directional derivative, and what is its interest.