Ph.D. candidates preliminary exam: Complex Variables **UC Merced, January 2008**

Directions: This examination lasts 4 hours.

- Partial credit will be awarded to relevant work.
- No credit will be awarded for unexplained answers, correct or not.
- Computations mistakes will be very lightly penalized.
- Accurate graphic representation of a problem will receive high consideration.
- All questions are worth the same number of points.

1. Find all complex numbers z such that the following equalities hold AND indicate where these points are on the complex plane.
   (a) $z = i^{1/2}$
   (b) $z = \log(-2)$
   (c) $\bar{z} = 1/z$
2. Using Taylor Series (assume that they are converging), argue that Euler's formula, which relates complex exponentials to trigonometric functions, is true for all complex numbers z.
3. Sketch the surface representing the real function $f(z) = |e^{iz}|$. Use as your coordinates system $Re(z)$, $Im(z)$ and $f(z)$.
4. Determine the Taylor Series centered at $z = 1+i$ of $f(z) = \frac{1}{z}$ and determine its radius of convergence. From the Taylor Series, find $f^{(5)}(i+1)$.
5. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2-2x+2)^2}$.
6. Evaluate $\int_0^{\infty} \frac{dx}{x^{1/2}(x^2+1)}$ using Cauchy's Residue Theorem.
7. Express the inverse Laplace transform of $F(s) = \frac{1}{s^2 \cosh s}$, in terms of the residues of some complex function. Identify where the residues have to be evaluated and the order of the corresponding poles.
8. Sketch the following mappings applied to the specified domains. Identify where each boundary is mapped to.
   (a) $w = f(z) = \cos z$, applied to the half infinite strip $0 \le Re(z) \le \pi$, $Im(z) \ge 0$.
   (b) $w = f(z) = \frac{i}{z}$, applied to the upper half of the unit cricle centered at the origin.
9. Using a logarithmic mapping, find the steady temperature distribution (which is by definition a harmonic function) in the portion of annulus for which $x \ge 0$, $y \ge 0$ and $1 \le x^2 + y^2 \le e^2$ and satisfying the following boundary conditions

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\begin{aligned}
T(x,y) &= \log x && \text{if } 1 \le x \le e,\ y = 0 \\
T(x,y) &= \log y + \pi/2 && \text{if } x = 0,\ 1 \le y \le e \\
T(x,y) &= \arctan(y/\sqrt{1-y^2}) && \text{if } x^2 + y^2 = 1 \\
T(x,y) &= 1 + \arctan(y/\sqrt{1-y^2}) && \text{if } x^2 + y^2 = e^2
\end{aligned}
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