Applied Math Preliminary Exam: Linear Algebra

UC Merced, January 2009

Directions: This examination lasts 4 hours.

Problem 1) For each of the following matrices determine whether there exists a similarity transformation with a unitary matrix U which can transform these matrices into diagonal form. Your answer must be fully justified, no credit will be given for "yes"/"no" answer.

(a)

$$\begin{bmatrix} 2 & 3-3i & i \\ 3+3i & 5 & 1 \\ -i & 1 & -3 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

Problem 2) Given the system of equations:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= b_1 \\ x_1 + 2x_2 + 4x_3 + 6x_4 &= b_2 \\ x_3 + 2x_4 &= b_3, \end{aligned}$$

(a) find all possible values of b_1, b_2, b_3 for which this system has solutions;

(b) find all possible solutions of this system if $b_1 = 0, b_2 = 1, b_3 = -1$.

Problem 3) For which values of b the following matrix is positive-definite

$$\begin{bmatrix} b & 3 & 0 \\ 3 & b & 4 \\ 0 & 4 & b \end{bmatrix}$$

Problem 4) Does the system $Ax = b$ with

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

have a solution? If the answer is "yes" find this solution, if the answer is "no" find $\hat{x}$ such that the 2-norm of the error is minimized, i.e. $\|A\hat{x} - b\|_2 = \min_{x \in \mathbb{R}^4} \|Ax - b\|_2$.

Problem 5) Given

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

(a) Find QR factorization of matrix

(b) Determine an orthonormal basis which spans the column space of A.

Problem 6) Prove that all eigenvalues of a Hermitian matrix are real.

Problem 7) Suppose A is a 2 by 2 symmetric matrix with unit eigenvectors u_1 and u_2. If its eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$, what are the matrices U, Σ and V^T that give singular-value decomposition of A?

Problem 8) Explain whether or not the following matrices can be similar to each other (your answers must be justified, "yes"/"no" answers will not get any credit):

(a) a symmetric matrix and a non-symmetric matrix,

(b) an invertible matrix and a singular matrix,

(c) an arbitrary matrix and an upper-triangular matrix.

Problem 9) Show that if A is an orthogonal matrix then $det(A)$ is either 1 or -1.

Problem 10) Suppose that you know that matrix A with columns consisting of vectors v_1, v_2, v_3 is invertible. Will a matrix with columns $w_1 = v_2 - v_3$, $w_2 = v_1 - v_3$ and $w_3 = v_1 - v_2$ be invertible? Justify your answer, no credit will be given for "yes"/"no" answer.