Applied Math Preliminary Exam: Linear Algebra

University of California, Merced, January 2010

Instructions: This examination lasts 4 hours.

- Show explicitly steps and computations in your solutions. Credit will not be given to answers without explanation.
- Partial credit will be awarded to relevant work.

Problem 1. Given the system of linear equations

$$\begin{array}{rcrcrcrcl} x_1 & - & x_2 & + & x_3 & + & x_4 & = & b_1 \\ -x_1 & + & x_2 & & & & & = & b_2 \\ -2x_1 & + & 2x_2 & - & x_3 & - & x_4 & = & b_3 \end{array}$$

a. (5 points) find all possible values of b_1, b_2, and b_3 for which this system has solutions;

b. (5 points) find all possible solutions of this system if $b_1 = -10$, $b_2 = 8$, and $b_3 = 18$.

Problem 2. (10 points) Let $A \in \mathbb{R}^{4\times 4}$ be a product of two matrices given by

$$A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 2 & 0 & 0 \end{bmatrix}.$$

Find dimensions and bases for Range(A), Null(A), Range(A^T), and Null(A^T).

Problem 3. (10 points) Prove that if $A \in \mathbb{R}^{n\times n}$ is symmetric, positive definite, and orthogonal, then A must be the identity matrix. (Hint: Consider the eigenvalues of A.)

Problem 4. Let $x \in \mathbb{R}^n$ and $A = uv^T$ where $u, v \in \mathbb{R}^n$.

a. (5 points) Prove that $\|x\|_\infty \le \|x\|_2 \le \|x\|_1$.

b. (5 points) Prove that $\|A\|_2 = \|u\|_2\|v\|_2$.

Problem 5. Let H be the Householder transformation given by

$$H = I - \frac{2}{\|u\|_2^2} uu^T,$$

where $u \in \mathbb{R}^n$.

a. (5 points) Show that H is symmetric and orthogonal.

b. (5 points) Determine the eigenvalues and the corresponding eigenvectors of H.

c. (5 points) Let $x \in \mathbb{R}^n$ have unit length, i.e., $\|x\|_2 = 1$, and let e_1 be the first column of the $n \times n$ identity matrix, i.e., $e_1 = (1, 0, 0, \cdots, 0)^T$. Show that if $u = x - e_1$, then

$$Hx = e_1.$$

Problem 6. Solve the least-squares problem

$$\min_{x \in \mathbb{R}^3} \|Ax - b\|_2$$

where

$$A = \begin{bmatrix} 2 & 7 \\ -1 & -5 \\ -2 & -4 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}.$$

using

a. (5 points) the normal equation,

b. (10 points) the QR factorization.

Verify that your solutions agree.

Problem 7. Let $A \in \mathbb{R}^{m \times n}$ have rank n.

a. (5 points) Show that $P = A(A^TA)^{-1}A^T$ is a projection matrix.

b. (10 points) By considering the singular value decomposition of A, show that

$$\|A\left(A^TA\right)^{-1}A^T\|_2 = 1.$$

Problem 8. Give an example for each of the following. Briefly explain why.

a. (3 points) A matrix that is positive definite but not symmetric.

b. (3 points) A non-diagonalizable matrix.

Problem 9. State whether each of the following statements is true or false. Briefly explain why.

a. (3 points) If $y \in \mathbb{R}^n$ is not in the null space of $A \in \mathbb{R}^{m \times n}$, then it must be in the range space of A^T.

b. (3 points) Any underdetermined linear system $Ax = b$, where $A \in \mathbb{R}^{m \times n}$ with $m < n$ and $b \in \mathbb{R}^m$, has infinitely many solutions.

c. (3 points) If A is an invertible matrix, then the inverse of its transpose is the same as the transpose of its inverse, i.e.,

$$(A^T)^{-1} = (A^{-1})^T.$$