Applied Math Preliminary Exam: Linear Algebra

University of California, Merced, January 2011

Instructions: This examination lasts 4 hours.

- Show explicitly steps and computations in your solutions. Credit will not be given to answers without explanation.
- Partial credit will be awarded to relevant work.

Problem 1. Let

$$A = \begin{bmatrix} 1 & -4 & 4 & 1 \\ 2 & -1 & 3 & 0 \\ -3 & -2 & -2 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix}.$$

a. (5 points) Determine the rank of A.

b. (5 points) Find *all* solutions to the system $Ax = b$, where $x \in I\!R^4$.

c. (5 points) Find a basis for Range(A), Null(A), Range(A^T), and Null(A^T).

d. (5 points) Verify that Range(A) is orthogonal to Null(A^T), i.e., if $w \in$ Range(A) and $v \in$ Null(A^T), then $w \perp v$.

Problem 2. (10 points) Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 3 & -1 & 4 \end{bmatrix}.$$

What are the eigenvalues and the corresponding eigenvectors of A? Is A diagonalizable? Why or why not?

Problem 3. Let I be the $n \times n$ identity matrix, and let $u \in I\!R^n$ have unit norm, $\|u\|_2 = 1$.

a. (5 points) Show that $M = I - uu^T$ is a projection matrix.

b. (5 points) What are the eigenvalues and the corresponding eigenvectors of M? Explain why.

c. (5 points) Onto which subspace of $I\!R^n$ is M the projection? Explain why.

Problem 4. (10 points) Let $A \in I\!R^{n \times n}$ with the following singular value decomposition:

$$A = U \Sigma V^T,$$

where the matrices U, Σ, and $V \in I\!R^{n \times n}$ where U and V are orthogonal and $\Sigma = \mathrm{diag}(\sigma_1^2, \cdots, \sigma_n^2)$ with $\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_m^2 \geq 0$. If A has rank $0 < r < n$, what do the columns of U and V represent? Explain why.

Problem 5. Let $A \in I\!R^{n \times n}$ be nonsingular and let x, y be vectors in $I\!R^n$.

(a) (5 points) Prove that if $(1 + y^T A^{-1} x) = 0$, then $(A + xy^T)$ is singular.

(b) (10 points) Prove that if $(1 + y^T A^{-1} x) \neq 0$, then the inverse of $A + xy^T$ is given by

$$(A + xy^T)^{-1} = A^{-1} - \frac{1}{1 + y^T A^{-1} x} A^{-1} x y^T A^{-1}$$

Problem 6. (10 points) Does the system $Ax = b$ with

$$A = \begin{bmatrix} 7 & 3 \\ 0 & 2 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 4 \\ 1 \\ -3 \\ 2 \end{bmatrix}$$

have a solution? If the answer is "yes", find this solution. If not, find $\hat{x}$ such that the 2-norm of the error is minimized, i.e., $\|A\hat{x} - b\|_2 = \min_{x \in I\!R^4} \|Ax - b\|_2$.

Problem 7. Short answers.

a. (5 points) Prove that a matrix $A \in I\!R^{n \times n}$ whose rows each sum to zero is singular.

b. (5 points) Show that the matrix

$$B = \begin{bmatrix} 1 & -5 \\ 0 & 2 \end{bmatrix}$$

has positive eigenvalues but is not positive definite.

c. (5 points) On separate graphs, draw the following sets:

(a) $F = \{x \in I\!R^2 \colon \|x\|_1 = 1\}$.

(b) $G = \{x \in I\!R^2 \colon \|x\|_2 = 1\}$.

(c) $H = \{x \in I\!R^2 \colon \|x\|_\infty = 1\}$.

d. (5 points) For $x \in I\!R^n$, show that the operator $\|x\|_0 \equiv \{\# \text{ of nonzeros in } x\}$ is not a norm.