Applied Math Preliminary Exam: Linear Algebra

University of California, Merced, January 2017

Instructions: This examination lasts 4 hours. Show explicitly steps and computations in your solutions. Credit will not be given to answers without explanation. Partial credit will be awarded to relevant work. The total number of points is 100.

Problem 1. (10 points) Suppose $v_1, v_2, \ldots, v_k$ are orthogonal vectors.

(a) Prove that $v_1, v_2, \ldots, v_k$ are linearly independent.

(b) Suppose $u \in \operatorname{span}(v_1, v_2, \ldots, v_k)$. Prove that u is uniquely written as a linear combination of the v_i's.

Problem 2. (10 points) Which of the following sets are vector spaces? Define a basis for the sets that are vector spaces.

(a) $\mathcal{S}_1 = \{A \in \mathbb{R}^{2\times 2} \colon A \text{ is upper triangular}\}$

(b) $\mathcal{S}_2 = \{A \in \mathbb{R}^{2\times 2} \colon A \text{ is symmetric}\}$

(c) $\mathcal{S}_3 = \{A \in \mathbb{R}^{2\times 2} \colon A \text{ is nonsingular}\}$

Problem 3. (10 points) Find an orthogonal basis for the space spanned by

$$\left\{ \begin{bmatrix} 8 \\ 3 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix} \right\}.$$

Problem 4. (10 points) Consider the system of linear equations

$$\begin{array}{rcrcrcrcl} & - & 2x_2 & + & 4x_3 & - & 6x_4 & = & b_1 \\ 5x_1 & + & x_2 & + & 3x_3 & + & 3x_4 & = & b_2 \\ 8x_1 & + & 3x_2 & + & 2x_3 & + & 9x_4 & = & b_3 \end{array}$$

(a) Find all possible values of b_1, b_2, and b_3 for which this system has solutions.

(b) Find all possible solutions of this system if $b_1 = -2$, $b_2 = 1$, and $b_3 = 3$.

Problem 5. (10 points) Let

$$A = \begin{bmatrix} 3 & -5 \\ 1 & 1 \\ -1 & 0 \\ 3 & -2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 2 \end{bmatrix}.$$

Find vectors $b_C \in \operatorname{Col}(A)$ and $b_N \in \operatorname{Null}(A^T)$ such that $b = b_C + b_N$.

Problem 6. (10 points) Let I be the $n \times n$ identity matrix, $u \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$.

(a) What are the eigenvalues and corresponding eigenvectors of $I + \alpha u u^T$?

(b) Derive the value(s) of α so that $I + \alpha u u^T$ is a projection matrix.

Problem 7. (10 points) Let $x \in \mathbb{R}^n$, and let $e_1 \in \mathbb{R}^n$ be the first column of the $n \times n$ identity matrix, i.e., $e_1 = [\,1 \;\; 0 \;\; 0 \;\; \cdots \;\; 0\,]^T$. Let

$$u = x - \|x\|_2 e_1.$$

(a) Compute $\|u\|_2^2$.

(b) Compute
$$\left(I - \frac{2}{\|u\|_2^2} uu^T\right) x.$$
(Hint: Your answer should be a multiple of e_1.)

Problem 8. (15 points) Let x and y be vectors in $\mathbb{R}^n$ such that $x \perp y$. Let $A = [\, x \quad y \,] \in \mathbb{R}^{n\times 2}$, i.e., A is the matrix whose columns are x and y.

(a) Compute the singular value decomposition of A.

(b) Find a basis for (i) the column space of A, (ii) the column space of A^T, (iii) the null space of A, and (iv) the null space of A^T.

Problem 9. (15 points) True or false. Provide a short explanation for each case.

(a) The product of two singular matrices is singular.

(b) If trace(A) > 0, then A is positive definite.

(c) The eigenvalues of real symmetric matrices are real.